

Student Name: _____

Mathematics Class: _____



James Ruse Agricultural High School

2024 YEAR 11 Task 3

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black pen
- Calculators approved by NESA may be used.
- NESA Reference sheet is provided.
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student number at the top of every page of your answer sheets.

Total marks: 53 marks

Section I – 5 marks

- Attempt Questions 1–5 on the Multiple Choice Answer Sheet.
- Allow about 8 minutes for this section

Section II – 48 marks

- Attempt Questions 6–9
- **Start EACH Question on a new sheet of paper.**
- Allow about 82 minutes for this section.

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Section I

5 marks.

Attempt Questions 1-5.

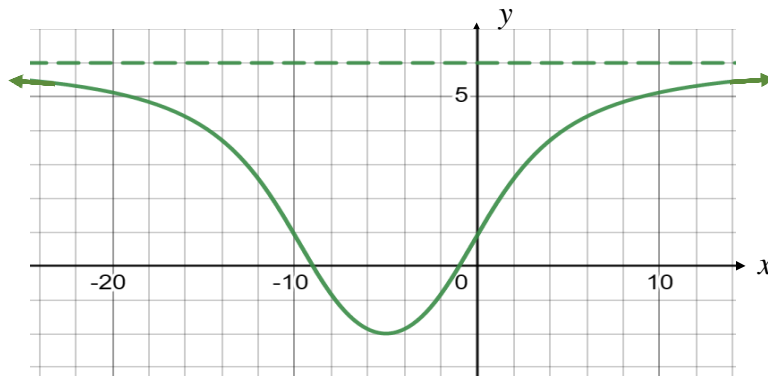
Allow about 8 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-5.

1 Which of the following is the value of $\cos^{-1}(\cos a)$ if $\pi < a < \frac{3\pi}{2}$?

- A. a
- B. $-\pi + a$
- C. $2\pi - a$
- D. $-2\pi + a$

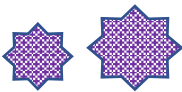
2 The diagram shows the graph of the function $y = f(x)$.



What is the range of $y = \frac{1}{f(x)}$?

- A. $(-\infty, -\frac{1}{2}] \cup (\frac{1}{6}, \infty)$
- B. $(-\infty, -2) \cup [\frac{1}{6}, \infty)$
- C. $(-\infty, -2) \cup [6, \infty)$
- D. $(-\infty, -\frac{1}{2}) \cup (\frac{1}{6}, \infty)$

- 3 In how many ways the letters of the word NALGONDA can be arranged if the letters G, O and D occur in the same order (G before O and O before D)?
- A. 840
 - B. 1260
 - C. 1680
 - D. 6720

- 4 Each one of these two stars  is to be placed at random on a vertex of an octagon.

What is the probability that these two stars will be placed on a diagonal?

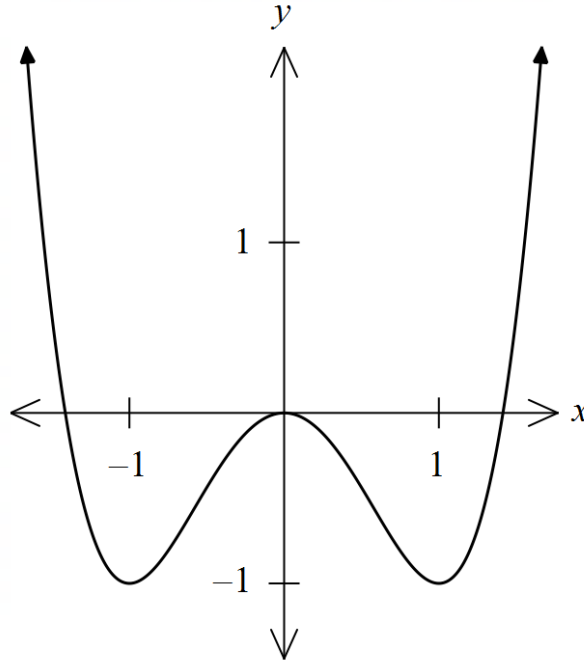
- A. $\frac{5}{7}$
- B. $\frac{1}{4}$
- C. $\frac{1}{10}$
- D. $\frac{1}{14}$

5

The function $f(x) = -\sqrt{1-\sqrt{1+x}}$ has an inverse $f^{-1}(x)$.

The graph of $y = f^{-1}(x)$ forms part of the curve $y = x^4 - 2x^2$.

The diagram shows the curve $y = x^4 - 2x^2$.



How many points do the graphs of $y = f(x)$ and $y = f^{-1}(x)$ have in common?

- A. 1
- B. 2
- C. 3
- D. 4

End of Section I

Section II

48 marks.

Attempt Questions 6-9.

Allow about 82 minutes for this section.

Start each question on a new piece of paper.

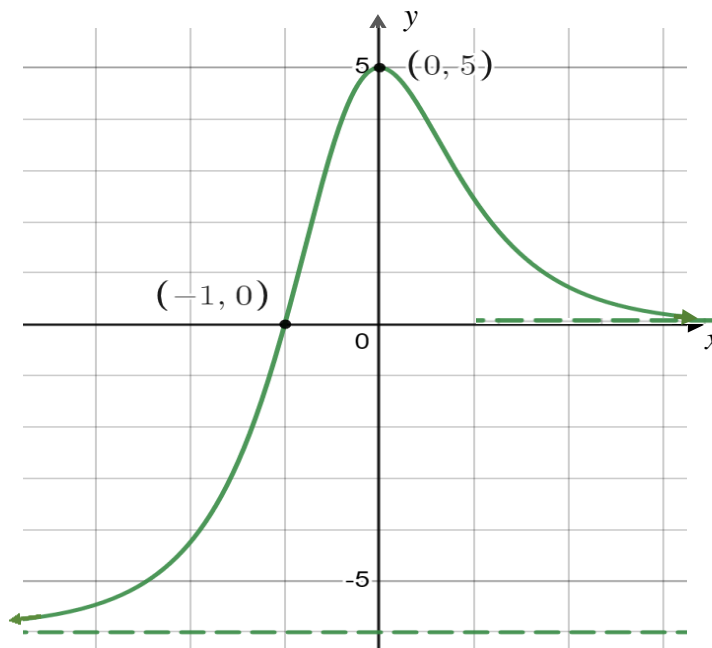
Question 6 Begin answer on a new piece of paper

11 Marks

(a) Solve the inequality $\frac{3}{2x+5} - x \geq 0$.

3

(b) The diagram below shows the graph of $y = f(x)$.



On the separate page provided at the end of this paper, sketch the following transformations, showing all asymptotes and intercepts and submit it with the rest of your solutions to Question 6.

(i) $y = \sqrt{f(x)}$

1

(ii) $|y| = f(|x|)$

2

(c) By making the substitution $t = \tan \frac{\theta}{2}$, show that $\operatorname{cosec} \theta + \cot \theta = \cot \frac{\theta}{2}$

2

(d) A class of fifteen mathematics students are seated around a circular table to discuss mathematical problems. If the seats are randomly assigned, find the probability that four particular students, Sam, Toby, Euan and Will, are **not** all seated together as a group of four. Give your answer in the simplest form.

3

End of Question 6

Question 7 Begin answer on a new piece of paper**13 Marks**

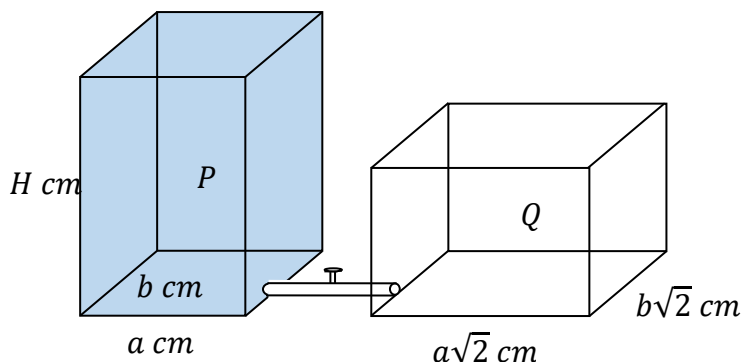
- (a) (i) Find the domain and range of $y = 2\cos^{-1}(x^2 - 3)$. **3**
- (ii) Hence, sketch the graph of $y = 2\cos^{-1}(x^2 - 3)$ showing all important features. **2**
- (b) The radius of a sphere is increasing at the rate of 0.05 cm s^{-1} . **2**
Find the rate at which its volume is increasing when the volume is $288\pi \text{ cm}^3$
- (c) The parametric equations of a curve are $x = \cos(2t)$ and $y = \sin(4t)$, $t \in [0, \pi]$.
- (i) Find the Cartesian equation of the curve. **2**
- (ii) Hence, sketch the curve on Cartesian number plane about one third of a page in size. **2**
- (d) The letters of the word **MONKEYS** are to be arranged in a row. **2**
Find the number of ways that the letters **K** and **E** will be separated by at least 2 letters.

End of Question 7

Question 8 Begin answer on a new piece of paper**12 Marks**

(a) Show that $\sin^{-1}\left(\frac{1}{\sqrt{1+e^{-2a}}}\right) + \tan^{-1}(e^a) = \tan^{-1}\left(\frac{2}{e^{-a}-e^a}\right)$, where a is a non-positive real number. **3**

(b) The diagram shows two rectangular prism tanks, P and Q connected by a small valve, with base dimensions a cm by b cm and $a\sqrt{2}$ cm by $b\sqrt{2}$ cm respectively.



Initially, tank P is full of liquid and tank Q is empty.

At time $t = 0$, the small valve opens and the liquid from tank P passes into tank Q . The height of liquid in tank Q increases at a rate proportional to the difference in the heights of liquid in the two tanks.

At a time t , the level of liquid in tank Q is h cm.

(i) Show that $\frac{dh}{dt} = k(H - 3h)$, where k is a positive constant. **3**

Hint: Let x be the drop in height of the water in tank P at time t .

(ii) Show that $h = \frac{H}{3}(1 - e^{-3kt})$ is a solution to the differential equation in part (i). **1**

(iii) After 4 seconds, the level of liquid in tank Q is 10% of H . **2**

Find, as a percentage of H (correct to the nearest percent), the level of liquid in tank Q after 16 seconds.

(c) Find the number of ways in which the letters of the word PERCEPTIVE can be arranged if:

(i) There are no restrictions. **1**

(ii) All the vowels are separated. **2**

End of Question 8

Question 9 Begin answer on a new piece of paper**12 Marks**

(a) (i) Using compound angle results, show that $\sin 3x = 3 \sin x - 4 \sin^3 x$. **3**

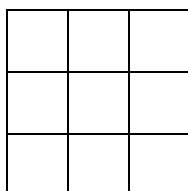
(ii) Consider the equation $\sin 2x = \sin 3x$.

You are given that $x = \frac{\pi}{5}$ is a solution to this equation.

Show that $x = \frac{3\pi}{5}$ is another solution to this equation. **1**

(iii) Hence, find the exact value of $\cos \frac{3\pi}{5}$. **3**

(b) A 3×3 grid is to be filled with numbers from the set $\{-1, 0, 1\}$, with repetition of the numbers allowed. **2**



Prove that among the sums, by rows, columns and diagonals, there are at least two of these sums that are equal.

(c) Shikung and Andrew are competing against each other in a competition in which the winner is the first to score five goals. The outcome is recorded by listing, in order, the initial of the person who scores each goal. For example, one possible outcome is recorded as SAASSASS.

(i) Explain why there are five different ways in which the outcome could be recorded if Andrew scores only one goal in the competition. **1**

(ii) In how many different ways could the outcome of this competition be recorded? **2**

End of Examination

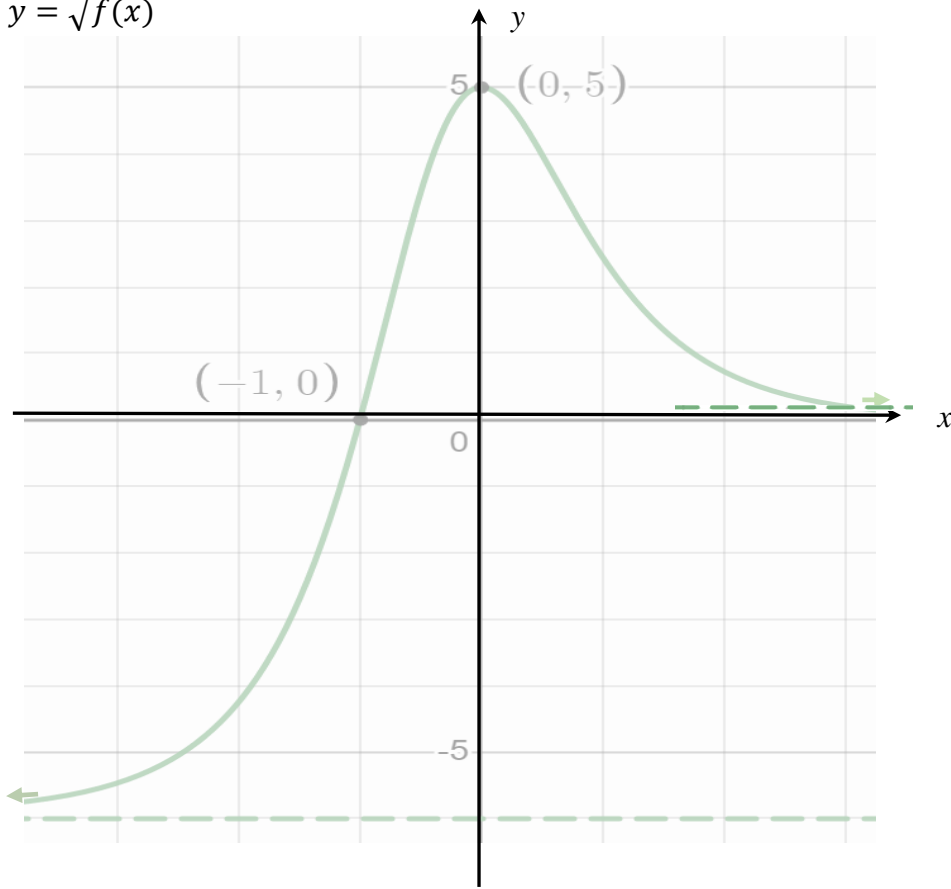
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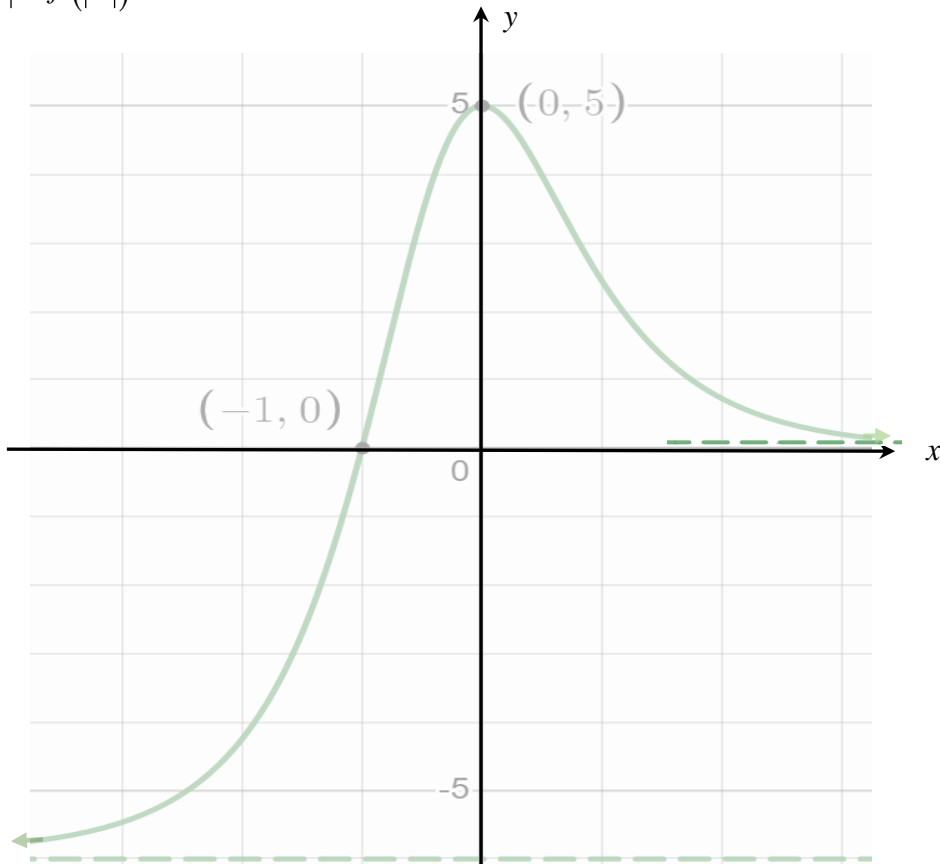
Class: _____

Answer Sheet for Question 6 (b). Detach and staple together with your solutions for Question 6.

(i) $y = \sqrt{f(x)}$



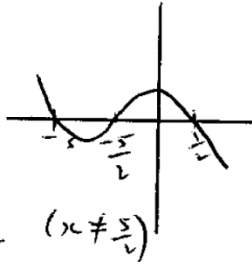
(ii) $|y| = f(|x|)$



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Question 6

6 a) $\frac{3}{2x+5} - x \geq 0 \quad x \neq -\frac{5}{2}$
 $3(2x+5) - x(2x+5)^2 \geq 0$
 $(2x+5)(3-5x-2x^2) \geq 0$
 $(2x+5)(x+3)(1-2x) \geq 0$
 $\therefore x \leq -3 \text{ or } -\frac{5}{2} < x \leq \frac{1}{2} \quad (x \neq -\frac{5}{2})$



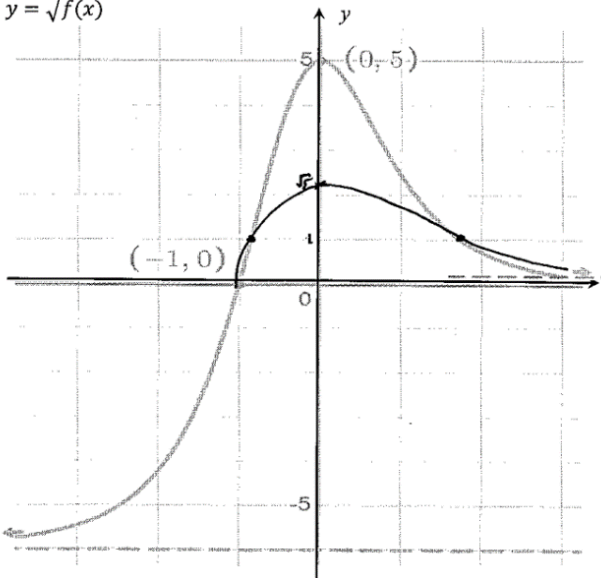
Ans 1 correct linear factors in inequality
 Ans 2 correct inequality without $x \neq -\frac{5}{2}$
 Ans 3 correct solution

1 mark if identified $x \neq -\frac{5}{2}$.

0 mark if not cubic equation and not identified $x \neq -\frac{5}{2}$

Number of students expanded $x(2x+5)^2$, used long division to factorise and spent more time solving the inequality.

b) (i) $y = \sqrt{f(x)}$



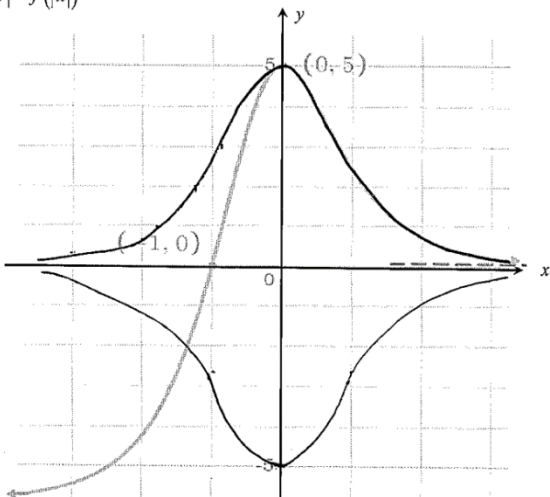
1 mark for the correct shape

Vertical tangent at $(-1, 0)$

Intersect $y = f(x)$ at $y = 1$

y -intercept $(\sqrt{5}, 0)$

ii) $|y| = f(|x|)$



1 mark for the correct shape for $y = f(|x|)$

Some forgot to graph $y = -f(|x|)$

$$(c) \text{L.H.S.} = \operatorname{cosec} \theta + \cot \theta$$

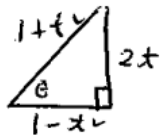
$$= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta}$$

$$= \frac{1 + \cos \theta}{\sin \theta}$$

$$= \frac{1 + \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} \times \frac{1+t^2}{1+t^2}$$

$$= \frac{1+t^2+1-t^2}{2t}$$

$$= \frac{2}{2t} = \frac{1}{t} = \frac{1}{\tan \frac{\theta}{2}} = \cot \frac{\theta}{2} = \text{R.H.S.}$$



Ans 1 Correct substitution of t -formula.

Ans 2 Correct algebraic progress to $\frac{1}{\tan \frac{\theta}{2}}$

Almost all did this part well!

d) Total number of seating arrangements
 $= (15-1)! = 14!$

Consider Sam, Toby, Euan and Will as one element \therefore we are arranging 12 elements around a circle $(12-1)!$ ways.

$$\therefore P(\bar{E}) = \frac{11! \times 4!}{14!} = \frac{1}{91}$$

$$P(E) = 1 - P(\bar{E})$$

$$= 1 - \frac{1}{91}$$

$$= \frac{90}{91}$$

Ans 1 $(15-1)!$

Ans 2 $P(\bar{E}) = \frac{1}{91}$

Ans 3 $\frac{90}{91}$

Mostly well done on this part. However, some misunderstood the question.

“Sam, Toby, Euan and Will, **are not all seated together as a group of four.**”

Question 7

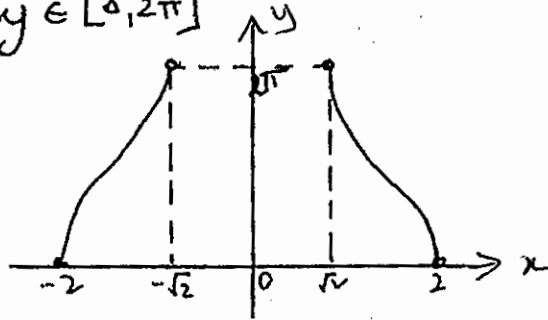
(a) i) $-1 \leq x^2 - 3 \leq 1 \Rightarrow 2 \leq x^2 \leq 4$

$\therefore x^2 \geq 2$ and $x^2 \leq 4$

$\therefore x \in [-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

$y \in [0, 2\pi]$

ii)



2 marks for
Correct domain
1 mark for
Correct range

Note: 1 mark for
partial correct
answer for
domain

2 marks for
Correct graph
1 mark for partial
Correct graph

(b) $\frac{dr}{dt} = 0.05 \text{ cm/s}$

When $V = \frac{4}{3} \pi r^3 = 288\pi$
 $r = 6$

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr} \times \frac{dr}{dt} \\ &= 4\pi r^2 \times 0.05 \\ &= 0.2\pi r^2 \end{aligned}$$

When $r = 6$, $\frac{dV}{dt} = 0.2\pi (6)^2$
 $= \frac{36\pi}{5} \text{ cm}^3/\text{s}$

1 mark for correct r

1 mark for final
Correct answer

c) i) $x = \cos 2t$ --- ①

$y = \sin 4t = 2 \cos 2t \sin 2t$

$\therefore y = 2x \sin 2t$

$\therefore \sin 2t = \frac{y}{2x}$ --- ②

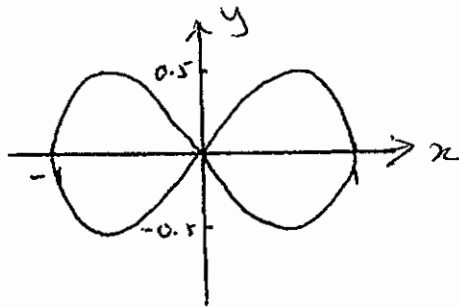
$\cos^2 2t + \sin^2 2t = 1$

$\therefore x^2 + \frac{y^2}{4x^2} = 1$

$y^2 = 4x^2(1-x^2)$

$\therefore y = \pm 2x\sqrt{1-x^2}$

ii)



2 mark for correct answer

1 mark for $y = 2x\sqrt{1-x^2}$

2 marks for correct graph and intercepts

1 mark for partial correct graph

d)

Method 1:

Number of ways with K and E together = $6! \cdot 2$

Number of ways with K and E and one letter in between = $5! \cdot 2 \cdot 5$

Number of ways with at least 2 letters between K and E = $7! - 6! \cdot 2 - 5! \cdot 2 \cdot 5 = 20 \cdot 5! = 2400$ ways

Method 2:

If E is the first letter from left in the arrangement, there are 4 positions for k to have at least 2 letters in between. If E is the Second letter from left in the arrangement, there are 3 positions for K. If E is the third Letter from left in the arrangement, there are 2 positions for K. If E is the fourth Letter from left in the arrangement, there is one position for k. You would have similar scenario with the letter K before the letter E from left. The number of arrangement of the other 5 letters is $5!$.

So the number of arrangement is $5! \cdot 2! \cdot (4+3+2+1) = 20 \cdot 5! = 2400$ ways.

2 marks for correct answer with explanation

1 mark for partial correct answer

Year 11 Extension 1 Task 3 (Yearly)

Suggested Solutions

Marks

Marker's Comments

Question 8.

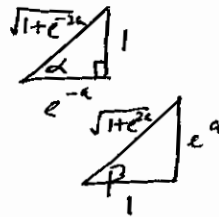
a) R.T.P $\sin^{-1}\left(\frac{1}{\sqrt{1+e^{-2a}}}\right) + \tan^{-1}(e^a) = \tan^{-1}\left(\frac{2}{e^{-a}-e^a}\right)$

ie, $\alpha + \beta = \gamma$

where $\sin \alpha = \frac{1}{\sqrt{1+e^{-2a}}}$

$\tan \beta = \frac{e^a}{1}$

$\tan \gamma = \frac{2}{e^{-a}-e^a}$



If $\alpha + \beta$ is to equal γ

R.T.P $\tan(\alpha + \beta) = \tan \gamma$

L.H.S = $\tan(\alpha + \beta)$

= $\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

= $\frac{\frac{1}{e^{-a}} + e^a}{1 - \frac{1}{e^{-a}} \cdot e^a} \times \frac{e^{-a}}{e^{-a}}$

= $\frac{1 + 1}{e^{-a} - e^a} = \frac{2}{e^{-a} - e^a} = \text{R.H.S}$

Aw1

A right angled triangle with either α or β and correct side lengths.

Aw2

A correct expression in terms of e via a correct use of a compound or double angle formula.

Aw3

detailed algebraic progress to result.

b) i) Decrease of volume in tank P
= Increase in volume in tank Q

$\therefore abx = 2abh$

$x = 2h$

Aw1

Liquid in tank P at time t

is $H - x = H - 2h$.

Aw2

$\therefore \frac{dh}{dt} \propto$ diff in Heights in P and Q

$\propto H - 2h - h$

Aw3

correct progress and notation to result.

$\therefore \frac{dh}{dt} = k(H - 3h)$.

Year 11 Extension 1 Task 3 (Yearly)

Suggested Solutions

Marks

Marker's Comments

b ii) If $h = \frac{H}{3}(1 - e^{-3kt})$

$$\begin{aligned} \frac{dh}{dt} &= \frac{H}{3} \times 3k e^{-3kt} \\ &= kH e^{-3kt} \quad \text{but } e^{-3kt} = 1 - \frac{3h}{H} \\ &= kH \left(1 - \frac{3h}{H}\right) \\ &= k \cancel{H} \times \frac{H-3h}{\cancel{H}} \\ &= k(H-3h) \end{aligned}$$

iii) when $t=4$, $h = \frac{1}{10}H$

$$\therefore \frac{1}{10}H = H(1 - e^{-12k})$$

Solving for k we have

$$k = -\frac{1}{12} \ln\left(\frac{7}{10}\right)$$

$$\text{when } t=16, h = \frac{H}{3} \left(1 - e^{-3 \times -\frac{1}{12} \ln\left(\frac{7}{10}\right) \times 16}\right)$$

$$\doteq 0.25H$$

ie, 25% of H

Aw1 correct differentiation and algebraic manipulation to $e^{-3kt} = 1 - \frac{3h}{H}$

Aw1 k value
Aw2 25% of H

8 c) i) $\frac{10!}{3!2!} = 302400$

ii) PERCEPTIVE \Rightarrow 10 elements 4 vowels E, E, E, I
6 consonants PRCP_T

- Consonants can be arranged $\frac{6!}{2!}$ ways.
- Arranging these 6 consonants creates 7 spaces to ensure that no two vowels are adjacent
- There a 7C_4 ways to fill 4 spaces for the vowels under the given restriction.
- within these 7C_4 groupings the four vowels can be arranged $\frac{4!}{3!}$ ways.

$$\therefore \frac{6!}{2!} \times {}^7C_4 \times \frac{4!}{3!} = 50400$$

Aw1 for correct answer.

Aw1 for

- $\frac{6!}{2!}$ or 7C_4 or $\frac{4!}{3!}$ with correct reasoning.
- 50400 without correct reasoning

Aw2

50400 with correct reasoning.

(iii)

$$\begin{aligned}\sin\left(2 \cdot \frac{3\pi}{5}\right) &= \sin\left(3 \cdot \frac{3\pi}{5}\right) \quad (\text{by (ii)}) \\ &= 3 \sin\left(\frac{3\pi}{5}\right) - 4 \sin^2\left(\frac{3\pi}{5}\right) \quad (\text{by (i)})\end{aligned}$$

so, by expansion on LHS:

$$2 \sin\left(\frac{3\pi}{5}\right) \cos\left(\frac{3\pi}{5}\right) = 3 \sin\left(\frac{3\pi}{5}\right) - 4 \sin^2\left(\frac{3\pi}{5}\right) \quad \dots (1)$$

and since $\sin\left(\frac{3\pi}{5}\right) \neq 0$, we have

$$2 \cos\left(\frac{3\pi}{5}\right) = 3 - 4 \left(1 - \cos^2\left(\frac{3\pi}{5}\right)\right) \quad \dots (2)$$

so, letting $c = \cos\left(\frac{3\pi}{5}\right)$, we have

$$c^2 - \frac{1}{2}c = \frac{1}{4}$$

$$\left(c - \frac{1}{4}\right)^2 = \frac{5}{16} \quad (\text{comp. square})$$

$$\therefore c = \frac{1 \pm \sqrt{5}}{4}$$

Now, $\frac{\pi}{2} < \frac{3\pi}{5} < \pi$, so $\cos\left(\frac{3\pi}{5}\right) < 0$, hence we reject the positive solution.

So,

$$\cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{4}$$

(iii)...this was a lot of work for one mark and not always successful.

First mark for establishing (1).

Students who did not substitute $\frac{3\pi}{5}$ immediately and stayed with the variable x , and who arrived at the situation $\sin x (4 \cos^2 x - 2 \cos x - 1) = 0$

1

and moved straight to $4 \cos^2 x - 2 \cos x - 1 = 0$

1

without dealing with the **logical** conclusion that $\sin x = 0$ also solves the equation, were penalised. You **cannot** ignore logical consequences of an argument because they are inconvenient.

Second mark was awarded for establishing (2) correctly.

Third mark awarded for deducing

1

$$\cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{4}$$

but students **must** have argued **why** the negative solution was indeed the case.

(b)

We have a 3×3 grid, each allotment allowing for one of the numbers $-1, 0, 1$ with repetition allowed. We have to consider all possible sums across rows, columns and diagonals. We see that the sums are bounded by $+3$ and -3 , and that it is possible to construct all sums in between: $-3, -2, -1, 0, 1, 2, 3$; that is, there are 7 possible sums. This is **ONE** such construction:

Entry	Entry	Entry	Sum
-1	-1	-1	-3
-1	-1	0	-2
-1	0	0	-1
0	0	0	0
0	0	1	1
0	1	1	2
1	1	1	3

Within the grid, there are 3 rows, 3 columns and 2 diagonals, making for 8 possible configurations over which to sum.

If we designate the sums as ‘pigeonholes’ and the configurations as ‘pigeons’, by the *pigeonhole principle*, there is at least one pigeonhole (sum) ‘containing’ at least $\left\lceil \frac{8}{7} \right\rceil = \left\lceil 1 + \frac{1}{7} \right\rceil = 2$ pigeons (configurations).

That is, there are at least 2 of the rows, columns or diagonals assigned to the same sum.

This problem was handled well by the majority of students.

First mark for establishing correctly **either** one of the ‘pigeons’ or ‘pigeonholes’.

Second mark awarded for arguing correctly to the conclusion.

1

1

(c)

Shikung (S) and Andrew (A) compete where the winner is the first to score five goals.

- (i) If A scores only one goal and S scores 5, we seek all arrangements of the six letters of the word $SSSSSA$, excluding the one word ending with A since in that instance, S has won, will not play. We have then

$$\frac{6!}{5!} - 1 = 5$$

ways in which S can win if A scores only one goal.

- (ii) We consider the number of ways in which S could win, and then double the result to include the number of ways A could win (by symmetry).

There is a direct way of performing the following calculation, though this method was the method adopted by most candidates, so it is the one given.

If S wins, S must be the last to score a goal. We must also have S winning 5 times. A , though, can win anywhere from 0 to 4 times (A cannot win 5 times here since only S winning overall is being considered).

We will generalise a structure. Let A win k times and S win 5 times. We consider arranging the $(5 + k)$ -letters

$$SSSSS \underbrace{A \cdots A}_k$$

where in every instance, the last letter is S :

$$\left[S, S, S, S, \underbrace{A \cdots A}_k \right] S$$

1

Part (i) was handled very well, no issues. Majority argued correctly. **Only one mark available.**

1

Part (ii) was completed well, too.

First mark awarded for a **reasonable** combinatorial argument (reasoning had to make sense, there needed to be **some** written argument, even though it may have been ultimately incorrect).

Second mark awarded for all correct reasoning. A lot of students answered this question correctly.

k	Number of words $\frac{(4+k)!}{4!k!} \times 1$
0	$\frac{4!}{4!} \times 1 = {}^4C_4$
1	$\frac{5!}{4!1!} \times 1 = {}^5C_4$
2	$\frac{6!}{4!2!} \times 1 = {}^6C_4$
3	$\frac{7!}{4!3!} \times 1 = {}^7C_4$
4	$\frac{8!}{4!4!} \times 1 = {}^8C_4$

Since each of the cases are disjoint, we have by the addition principle, the total number of ways that S may win as:

$$\sum_{i=4}^8 {}^iC_4 = {}^9C_5 \text{ (hockey – stick identity)}$$

$$= 126$$

The situation for A winning is symmetrical, so there are 126 ways in which A may win.

Hence in total, there are $2 \times 126 = 252$ ways in which a player may win.

1