Student Name: ____________________

Mathematics Class: ____________________

James Ruse Agricultural High School

2022 **YEAR 11 Task 3**

Mathematics Extension 1

Instructions:

- Total marks: 82.
- Reading time: 10 minutes.
- Working time: 120 minutes.
- Write using black pen.
- Calculators approved by NESA may be used.
- A NESA reference sheet is provided.
- Show all relevant mathematical reasoning or calculations.

Topic	**MC**	**11**	**12**	**13**	**14**	**Total**
Functions		a b c d /14		a b /8	a b /6	/28
Trigonometry		e /2	a /4		c /4	/10
Calculus			b c /7	c d /8		/15
Combinatorics		f g /2	d e /7	e /2	c d e /8	/19
MC						/10
Total	/10	/18	/18	/18	/18	/82

Section I

10 marks.
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

Question 1

A test comprises of 50 multiple choice questions. Each question may be answered by selecting one of A, B, C or D. Which of the following is the number of ways of completing the test?

(A) 50

(B) ${}^{50}C_4$

(C) ${}^{50}P_4$

(D) 4^{50}

Question 2

If $t = \tan\theta$, which of the following is equivalent to $2\operatorname{cosec} 2\theta$?

(A) $\frac{1+t^2}{t}$

(B) $\frac{1+t^2}{2t}$

(C) $\frac{t}{1+t^2}$

(D) $\frac{2t}{1+t^2}$

Question 3

Given

$$3n + 2 \leq 11 \quad \ldots (1)$$

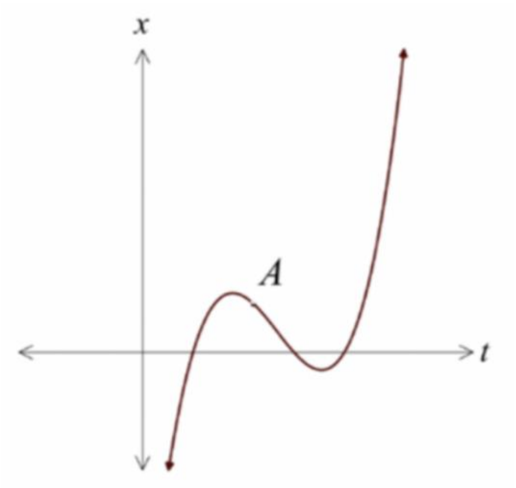
$$\frac{6n}{n^2 + 5} > 1 \quad \ldots (2)$$

which of the following gives all integer solutions n satisfying (1) and (2) simultaneously?

(A) 1, 2, 3, 4
(B) 2, 3, 4
(C) 1, 2
(D) 2, 3

Question 4

The diagram below shows the displacement, *x* metres, of a moving object at time *t* seconds.



Which of the following statements describes the motion of the object at the point A?

(A) Velocity is negative and acceleration is positive.
(B) Velocity is negative and acceleration is negative.
(C) Velocity is positive and acceleration is positive.
(D) Velocity is positive and acceleration is negative.

Question 5

Which of the following is the solution set to

$$\frac{1}{|x-2|} \geq 1 \text{ ?}$$

(A) $(1, 3)$

(B) $[1, 3]$

(C) $[1, 2) \cup (2, 3]$

(D) $(1, 2) \cup (2, 3)$

Question 6

Which of the following is the domain and range of $y = \sin^{-1}(\cos^{-1} x)$?

(A) Domain: $0 \leq x \leq 1$; Range: $0 \leq y \leq \frac{\pi}{2}$

(B) Domain: $\cos(1) \leq x \leq 1$; Range: $0 \leq y \leq \frac{\pi}{2}$

(C) Domain: $\cos(1) \leq x \leq 1$; Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(D) Domain: $-1 \leq x \leq 1$; Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Question 7

A bowl of fruit consists of two bananas, one apple, one pear and one mango. Assuming the bananas are identical, which of the following gives the number of ways of selecting two pieces of fruit?

(A) 7

(B) 10

(C) 15

(D) 20

Question 8

Which of the following inequalities has the same solution set as $|x+2|+|x-3|=5$?

(A) $\frac{5}{3-x} \geq 1$

(B) $\frac{1}{x-3} - \frac{1}{x+2} \leq 0$

(C) $x^2 - x - 6 \leq 0$

(D) $|2x-1| \geq 5$

Question 9

For which of the following conditions does the graph of

$$y = \frac{1}{ax^2 + bx + a} \quad \text{for } a \in \mathbb{R}\backslash\{0\}$$

have exactly two asymptotes?

(A) $-2 < \frac{a}{b} < 2$

(B) $-2 \leq \frac{a}{b} \leq 2$

(C) $-1 < \frac{b}{a} < 1$

(D) $\frac{a}{b} = \frac{1}{2}$

Question 10

Five contestants - Harry, Cedric, Viktor, Fleur and Tom - are in a race to find the centre of a maze. At the conclusion of the race, the following was known:

Harry beat Cedric.

Viktor beat Fleur.

Fleur beat Cedric.

Tom beat Cedric.

Assuming no ties, how many finishing orders could there have been?

(A) 8

(B) 12

(C) 16

(D) 20

Section II

72 marks.
Attempt Questions 11-14.
Allow about 1 hour and 45 minutes for this section.

Start each question on a new piece of paper.

Question 11 (18 marks) Begin answer on a new piece of paper.

(a) Solve 2

$$(1-x)^2 < \frac{9}{25}$$

(b) Solve 3

$$\big||2x+5|-3\big| = 8$$

(c) Consider

$$h(x) = \frac{1+2x}{7+x}$$

(i) Show that $h(x)$ is increasing for all x. 2

(ii) Explain why the result in part (i) ensures $h(x)$ will have an inverse without needing to restrict its natural domain. 1

(iii) Find $h^{-1}(x)$. 2

(d) Consider the function $f(x) = 3\sin^{-1}(x+1)$.

(i) Write down the domain and range of $f(x)$. 2

(ii) Sketch the graph of $y = f(x)$, giving the coordinates of the endpoints and any intercepts with the x- and y-axes. 2

(e) Evaluate $\cos 195° + \cos 105°$, leaving your answer in exact form. 2

Question 11 continues on page 8.

Question 11 (continued)

(f) What is the least number of students a class must have to guarantee at least 3 students share the same birthday month? Justify your answer. 1

(g) How many ‘words’ can be made from the letters of the word, ISOMORPHIC, if all letters are used exactly once? 1

End of Question 11.

Question 12 (18 marks) Begin answer on a new piece of paper.

(a) (i) Evaluate 2

$$\sin\left(\tan^{-1}\frac{3}{4}\right)$$

(ii) Show that 2

$$\cos\left(2\sin^{-1}\left(-\frac{3}{5}\right)\right) = \frac{7}{25}$$

(b) A particle moves in a straight line. Its displacement, t seconds, after leaving the fixed point O is x metres, where $x = \frac{1}{2}t^2 + \frac{1}{30}t^3$. Find the value of t for which the acceleration of the particle is twice its initial acceleration. 3

(c) A freshly caught fish, initially at 18°C, is placed in a freezer that has a constant unknown temperature of x^{o}C . The cooling rate of the fish is proportional to the difference between the temperature of the freezer and the temperature, T^{o}C , of the fish. It is known that the rate of change in the temperature of the fish is given by the equation

$$\frac{dT}{dt} = -k(T - x) \ \dots (*)$$

where t is the number of minutes the fish has been placed in the freezer.

(i) Show that $T = x + Ae^{-kt}$ satisfies equation $(*)$. 1

(ii) If the temperature of the fish is 10^{o}C at $7\frac{1}{2}$ minutes, show that the fish's temperature at t minutes is given by 3

$$T = x + (18 - x)e^{\frac{2}{15}\log_e\left(\frac{10-x}{18-x}\right)t}$$

Question 12 continues on page 10.

(d) A school tennis team comprises of 6 boys and 6 girls. Out of the 12 students, 4 boys and 4 girls are selected to join a round table dinner. Find the number of ways of arranging the table so that boys and girls alternate, but where Alexander and Catherine are seated together. Assume no one in the team shares the same name with anyone else. 3

(e) (i) A customer wishes to purchase a dozen donuts from a bakery that offers four different types of donut. The customer wishes to know how many ways there are of buying a dozen donuts. To do this, they represent each donut as a 'D' and place those twelve Ds in a line: 1

D D D D D D D D D D D D

They then insert three *s as place markers to indicate how many of each type of donut they buy.

For example: D D * D D D * D D D D D D * D D

means they buy 2 donuts of the first type, 3 donuts of the second type, 6 donuts of the third type and 2 donuts of the fourth type.

QUESTION: In how many ways can the customer buy 12 donuts if there are 4 available types? Assume there are at least 12 donuts available for each type.

(ii) Suppose now that there are only three types of donuts available: chocolate, strawberry and caramel. The customer wishes to select 10 donuts. In how many ways can this be done if only 5 strawberry donuts are available? Assume the other two types have at least 10 donuts available in each flavour. 3

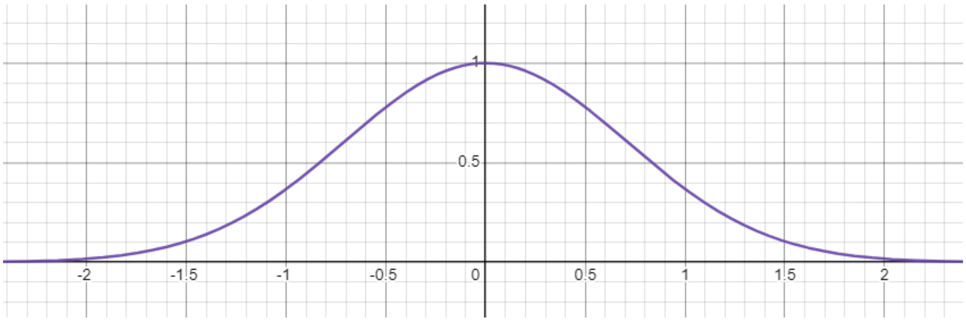
End of Question 12.

Question 13 (18 marks) Begin answer on a new piece of paper.

(a) Sketch $y = f(x)$ given **3**

$$f(x) = \frac{|x|}{x-1}$$

(b) Let $f(x) = e^{-x^2}$. The graph of $y = f(x)$ is shown below having domain $\mathbb{R}$ and range $0 < y \leq 1$.



(i) Find $f^{-1}(x)$ if the domain of $f(x)$ is restricted to all real $x \geq 0$. **2**

(ii) Sketch $y = f^{-1}(x)$ and $y = x$ on the same axes. **2**

(iii) Show that there is a solution to the equation **1**

$$x = e^{-x^2}$$

for $x \in [0.6, 0.7]$ (**do not** simply submit your graph in (ii) as your proof).

(c) A particle is moving such that, at time t seconds, its displacement, x metres, satisfies the equation **2**

$$t = 2 - 2e^{-3x}$$

Show that the particle always moves in the positive direction.

Question 13 continues on page 12.

(d) A given radioactive substance (the *parent* substance) decays into a stable form (the *daughter* substance). The amount of parent substance remaining after t years is given by

$$P(t) = Ae^{-kt}$$

where A and k are positive, real constants.

(i) Show that the rate of decay of the parent substance is directly proportional to the amount of parent substance remaining. 1

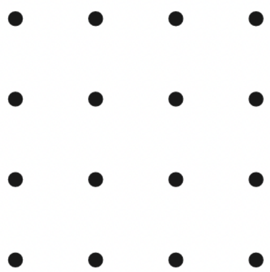
(ii) Determine, in terms of k, the time required for the activity to be halved. 2

(iii) In terms of $P(t)$ and A, write an expression for $D(t)$, the amount of daughter substance present at t years. 1

(iv) Hence show that 2

$$t = \frac{1}{k}\ln\left(\frac{D(t)}{P(t)} + 1\right)$$

(e) Sixteen nails are placed at equal intervals in the shape of a square as shown below. 2



How many triangles can be made using the nails as vertices?

End of Question 13.

Question 14 (18 marks) Begin answer on a new piece of paper.

(a) (i) Sketch 2

$$y = \frac{\pi}{2} - \cos^{-1} x$$

(ii) Let the curve $\mathcal{C}$ be defined by the parametric equations 4

$$x = \cos^{-1} t$$

$$y = \sin^{-1} t$$

By considering (i), or otherwise, find the Cartesian equation of $\mathcal{C}$, give the domain and range, and hence sketch the curve, showing all important features.

(b) Show that if $x \in \left(0, \frac{\pi}{4}\right)$, then 4

$$\tan^{-1}\left(\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right) = \frac{x}{2}$$

(c) Suppose 7 points lie on the circumference of a circle of radius 1 metre. Show that there must always be 2 points on the circumference such that the distance between them is less than 1 metre. Note, no two points may occupy the same location on the circumference. 3

(d) How many seven-letter words can be formed from the word, FOOLISH, if the L is always somewhere to the left of both Os and the S is always somewhere between both Os? 3

(e) Using a combinatorial argument, show that $(n^2)!/(n!)^n \in \mathbb{N}$ for all natural numbers n. 2

End of paper.

2022 Year 11 Extension Task 3

Multiple Choice Answers

1. D
2. A
3. D
4. B
5. C
6. B
7. A
8. C
9. D
10. B

Question 11:

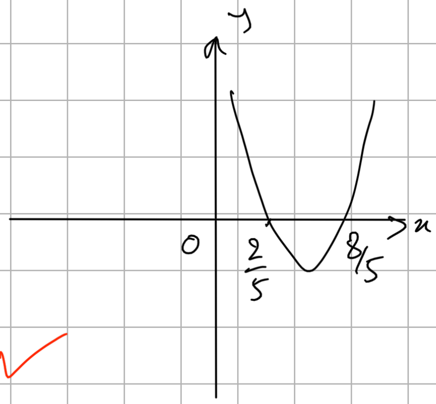
(a)

$$(1-x)^2 < \frac{9}{25}$$

$$(1-x)^2 - \left(\frac{3}{5}\right)^2 < 0$$

$$\left(1 - x + \frac{3}{5}\right)\left(1 - x - \frac{3}{5}\right) < 0$$

$$\left(\frac{8}{5} - x\right)\left(\frac{2}{5} - x\right) < 0$$ ✓



$$\frac{2}{5} < x < \frac{8}{5}$$ ✓

(b)

$$\big|\,|2x+5| - 3\,\big| = 8$$

$$\therefore\ |2x+5| - 3 = \pm 8$$ ✓

$$|2x+5| = 3 \pm 8$$

$$= 11 \text{ or } -5$$

but $|2x+5| > 0$ $\therefore$ $|2x+5| = 11$ ✓

$$\therefore\ 2x + 5 = \pm 11$$

$$2x = -5 \pm 11$$

$$x = \frac{1}{2}(-5 \pm 11)$$

$$x = -8 \text{ or } x = 3$$ ✓

Note: alternative methods

- starting with $|2x+5|$ with cases you need to be careful, because for each case, you need to eliminate solutions not in the domain of the case.

- Graphical method is accepted in the condition that full explanation and reference to the graph (graph must be labeled and all intercepts are shown clearly).

– providing extra solutions would result on losing the last mark.

(c) $h(x) = \dfrac{1+2x}{7+x}$

$= \dfrac{2(7+x) - 13}{7+x}$

$= 2 - \dfrac{13}{7+x}$

(i) $h'(x) = \dfrac{13}{(7+x)^2} > 0$ for all x ✓

$\therefore$ $h(x)$ is increasing for all x ✓

(ii) $h(x)$ is one to one function in $\mathbb{R}$ ✓

(iii) $y = \dfrac{1+2x}{7+x}$

$7y + xy = 1 + 2x$ ✓

$7y - 1 = x(2-y)$

$x = \dfrac{7y-1}{2-y}$

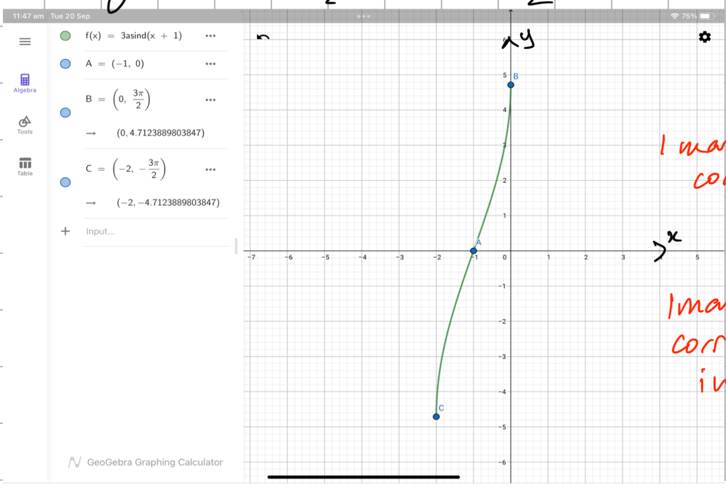
$\therefore h^{-1}(x) = \dfrac{7x-1}{2-x}$ ✓

(d) $f(x) = 3\sin^{-1}(x+1)$

(i) Domain: $-1 \le x+1 \le 1$

$-2 \le x < 0$ ✓

Range: $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}$ ✓



1 mark for the correct shape

1 mark for all correct intercepts.

(e) $\cos(195) + \cos(105)$

$= 2\cos\left(\frac{195+105}{2}\right)\cos\left(\frac{195-105}{2}\right)$ — 1 for correct steps

$= 2\cos(150)\cos(45)$

$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2}$ — 1 for correct answer

(f) 25 students ✓

$(2 \times 24 + 1)$

(g) $\frac{10!}{2!\,2!} = 907200$ ✓

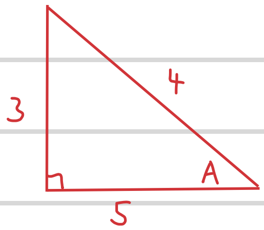
Year 11 Task 3 MATHEMATICS Ext 1 Question 12

Suggested Solutions	Marks	Marker's Comments

a) i let $A = \tan^{-1}\frac{3}{4}$

$\tan A = \frac{3}{4}$

$\therefore A \in [0, \frac{\pi}{2})$



1 mark for getting to tan A = 3/4 and constructing the triangle correctly

$\therefore \sin(\tan^{-1}\frac{3}{4}) = \sin A$

$= \frac{3}{5}$

1 mark for: reaching the final answer

ii let $A = \sin^{-1}(\frac{-3}{5})$

$\therefore \sin A = \frac{-3}{5}$ $\quad \therefore A \in [\frac{-\pi}{2}, 0]$

$\therefore \cos(2\sin^{-1}(\frac{-3}{5}))$

$= \cos 2A$

$= \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 1 - 2(\frac{-3}{5})^2$

$= 1 - \frac{18}{25}$

$= \frac{7}{25}$

1 mark for correctly expanding cos 2A

1 mark for completing the proof.

If students quote $\cos A = \frac{4}{5}$ without acknowledging that A is in the 4th quadrant, then they do not receive full marks

Year 11 Task 3 MATHEMATICS Ext 1 Question 12

Suggested Solutions	Marks	Marker's Comments

Alternatively, you can try remove the negative sign first.

$$\cos\left(2\sin^{-1}\left(\frac{-3}{5}\right)\right) = \cos\left(-2\sin^{-1}\left(\frac{3}{5}\right)\right) \quad \left(\sin^{-1}(-x) = -\sin^{-1}x\right)$$

$$= \cos\left(2\sin^{-1}\left(\frac{3}{5}\right)\right) \quad \left(\cos(-x) = \cos x\right)$$

1 mark for either:

- Removing the negative sign successfully with reason

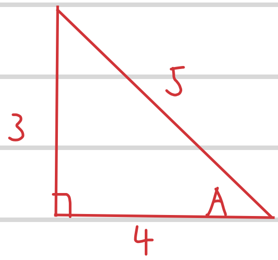
or

- Demonstrate the expansion of cos 2A

let $A = \sin^{-1}\frac{3}{5}$

$\sin A = \frac{3}{5}$

$\therefore A \in \left[0, \frac{\pi}{2}\right]$



$\therefore \cos\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

$= \cos 2A$

$= \cos^2 A - \sin^2 A$

$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$

$= \frac{16}{25} - \frac{9}{25}$

$= \frac{7}{25}$

1 mark for: Completing the proof.

Year 11 Task 3 MATHEMATICS Ext1 Question 12

Suggested Solutions	Marks	Marker's Comments
b) $x = \frac{1}{2}t^2 + \frac{1}{30}t^3$		
$\dot{x} = t + \frac{1}{10}t^2$		**1 mark for getting the velocity function**
$\ddot{x} = 1 + \frac{1}{5}t$		
when $t=0$, $\ddot{x} = 1 + \frac{1}{5}(0)$		**1 mark for getting the acceleration function**
$= 1$		
Solve $1 + \frac{1}{5}t = 2(1)$		
$= 2$		
$\frac{1}{5}t = 1$		
$t = 5$		**1 mark for final answer**
$\therefore$ after 5 seconds		
c) i $T = x + Ae^{-kt}$		
$\frac{dT}{dt} = -kAe^{-kt}$		**MUST show** $x + Ae^{-kt} - x$
$= -k(x + Ae^{-kt} - x)$		**or state that since T - x =** Ae^{-kt}
$= -k(T-x)$		
$\therefore T = x + Ae^{-kt}$ satisfies $\frac{dT}{dt} = -k(T-x)$		
ii When $t=0$, $T=18$		
$\therefore 18 = x + Ae^{-k(0)}$		
$= x + A$		**1 mark for solving for A**
$\therefore A = 18 - x$		
$\therefore T = x + (18-x)e^{-kt}$		
continue next page		

Year 11 Task 3 MATHEMATICS Ext 1 Question 12

Suggested Solutions	Marker's Comments
When $t = \frac{15}{2}$, $T = 10$	
$\therefore 10 = x + (18-x)e^{-\frac{15k}{2}}$	
$(18-x)e^{-\frac{15k}{2}} = 10 - x$	1 mark for making $e^{-\frac{15k}{2}}$ the subject
$e^{-\frac{15k}{2}} = \frac{10-x}{18-x}$	
$-\frac{15k}{2} = \log_e\left(\frac{10-x}{18-x}\right)$	
$-k = \frac{2}{15}\log_e\left(\frac{10-x}{18-x}\right)$	1 mark for completing the proof
$\therefore T = x + (18-x)e^{\frac{2}{15}\log_e\left(\frac{10-x}{18-x}\right)t}$	
d) First select 3 boys and 3 girls from the remaining 5 boys and 5 girls since Alexander and Catherine must be chosen. This can be done in $\binom{5}{3} \times \binom{5}{3}$ ways.	1 mark for getting the expression $\binom{5}{3} \times \binom{5}{3}$ with <u>clear explanation</u> If you failed to consider the initial selection process you will not receive anymore than 1/3 in total. (Fatal error)
Then we place Alexander and Catherine next to each other first in the circle, this can be done in 2 different ways.	
After that, we are arranging the other 3 boys and 3 girls in a line in alternating positions. This can be done $3! \times 3!$ ways.	1 mark for getting the expression $3! \times 3!$ with <u>clear explanation</u>
Therefore the total is $\binom{5}{3}^2 \times (3!)^2 \times 2 = 7200$	1 mark for getting to the final answer, but you <u>DO NOT</u> get a mark just for multiplying by 2 at some stage in the working
or	
First we place Alexander and Catherine next to each other first in the circle, this can be done in 2 different ways.	
Then we place 3 boys and 3 girls to from the remaining 5 boys and 5 girls into the circle, which is now a line because we have fixed Alexander and Catherine. This can be done in ${}^5P_3 \times {}^5P_3$ ways.	2 marks for getting the expression $({}^5P_3)^2$ with clear explanation.
Therefore the total is $2 \times ({}^5P_3)^2 = 7200$ ways.	1 mark for getting to the final answer, but you DO NOT get a mark just for multiplying by 2 at some stage in the working

Suggested Solutions	Marks	Marker's Comments

e) i. First condition to be clear, it is NOT necessary to buy at least one of each type.

The first * can be placed in 13 different positions

D D D D D D D D D D D D

The second * can be placed in 14 different positions after the first has been placed.

D D D D D D * D D D D D D

1 mark for answer

The third and final * can be placed in 15 different positions after that.

D D D D D D * D D D * D D D

However, if the 3 dividers are put in the same 3 positions but in different order, the same combination of donuts are still purchased.

Therefore the total is given by N = (13 x 14 x 15) ÷ 3!

= 455

or

This can be considered as arranging 15 objects in a line, with the 12 Ds being identical and 3 *s being identical.

Therefore the total number of ways this could be done is given by

N = 15! ÷ (12!3!)

= 455

1 mark for answer

D D * D D D D * D D D * D D D

Year 11 Task 3 MATHEMATICS Ext 1 Question 12

Suggested Solutions	Marker's Comments

ii. **We take cases:**

Case 1: 0 strawberries

Insert * in any of the 11 spaces among the donuts (there are 10 Ds)

↓D↓D↓D↓D↓D↓D↓D↓D↓D↓D↓

Case 2: 1 strawberry donuts

Insert * in any of the 10 spaces (only 9 Ds left)

↓D↓D↓D↓D↓D↓D↓D↓D↓D↓

Case 3: 2 strawberry donuts

Insert * in any of the 9 spaces (only 8 Ds left)

↓D↓D↓D↓D↓D↓D↓D↓D↓

continue the cases until we get to 5 strawberry donuts...

Case 6: 5 strawberry donuts

Insert * in any of the 6 spaces (only 5 Ds left)

↓D↓D↓D↓D↓D↓

The total number of ways this could be done would be:

N = 11 + 10 + 9 + 8 + 7 + 6

= 51 ways

Students receive:

1/3 if they recognise the need to sort into cases AND a systematic approach on the number of ways each case.

2/3 if they have considered all the cases and came up with an number close to the answer and the marker can clearly see where the minor mistake was made

3/3 if students considered all cases correctly and came up with the correct answer.

Year _____ Task _______ MATHEMATICS ______ Question______

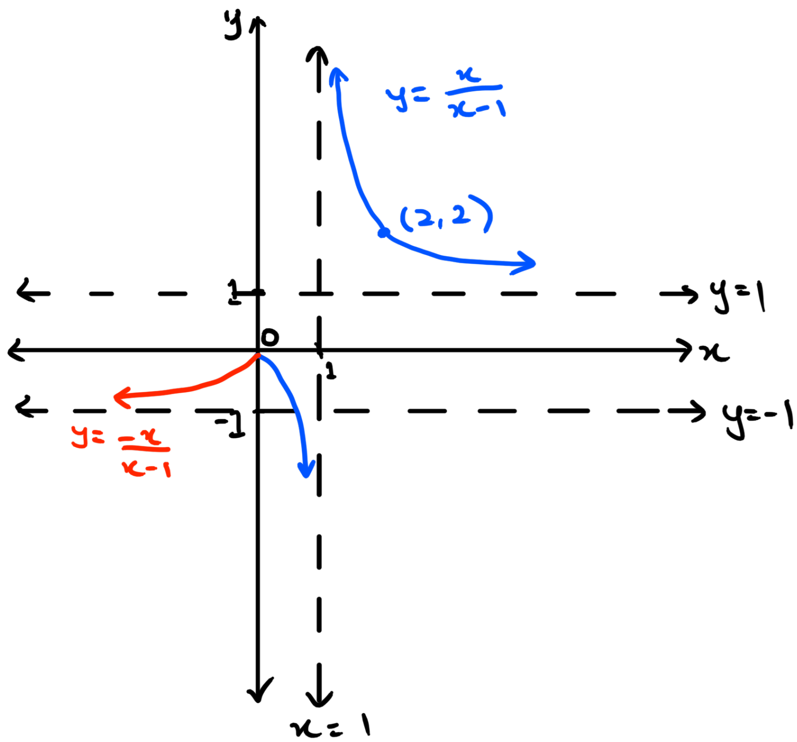
Suggested Solutions	Marks	Marker's Comments

Year _____ Task _______ MATHEMATICS ______ Question______

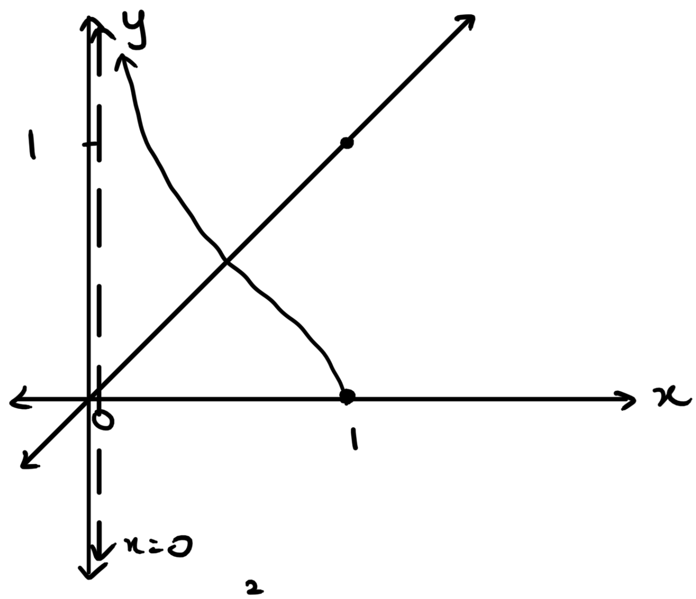
Suggested Solutions	Marks	Marker's Comments

Year _____ Task _______ MATHEMATICS ______ Question______

Suggested Solutions	Marks	Marker's Comments

MATHEMATICS Extension 1 : Question 13		
Suggested Solutions	**Marks**	**Marker's Comments**
a) $f(x) = \dfrac{\lvert x\rvert}{x-1}$ $= \begin{cases} \dfrac{x}{x-1}, & x \geq 0 \\ \dfrac{-x}{x-1}, & x < 0 \end{cases}$ 		1 must have cusp at (0,0) 1 shape 1 asymptotes
b) i) $f(x) = e^{-x^2}$ $x = e^{-[f^{-1}(x)]^2}$ $\ln x = -[f^{-1}(x)]^2$ $-\ln x = [f^{-1}(x)]^2$ $f^{-1}(x) = \pm\sqrt{-\ln x}$ range of f(x) : $0 < y \leq 1$ ∴ domain of f(x) : $0 < x \leq 1$ ∴ $\ln x \leq 0$ hence $-\ln x \geq 0$	1	

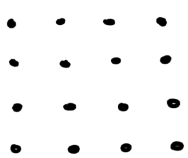
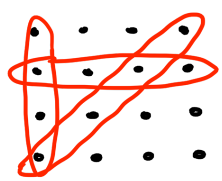
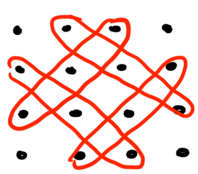
MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
domain of $f(x)$: $x \geq 0$ $\therefore$ range of $f^{-1}(x)$: $y \geq 0$ $\therefore$ $f^{-1}(x) = \sqrt{-\ln x}$	1	with explanation
ii) 		1 two correct curves 1 point of intersect within $0<x<1$ + asymptote
iii) $x = e^{-x^2}$ $\therefore e^{-x^2} - x = 0$ let $f(x) = e^{-x^2} - x$ $f(0.6) = e^{-0.6^2} - 0.6$ $= 0.0977$ (4 dp) $f(0.7) = e^{-0.7^2} - 0.7$ $= -0.0874$ (4dp) as $f(x)$ is continuous as e^{-x^2} and x are continuous, then there must be a value between 0.6 and 0.7 which will give $f(x) = 0$ $\therefore$ a solution exists between 0.6 and 0.7.	1	

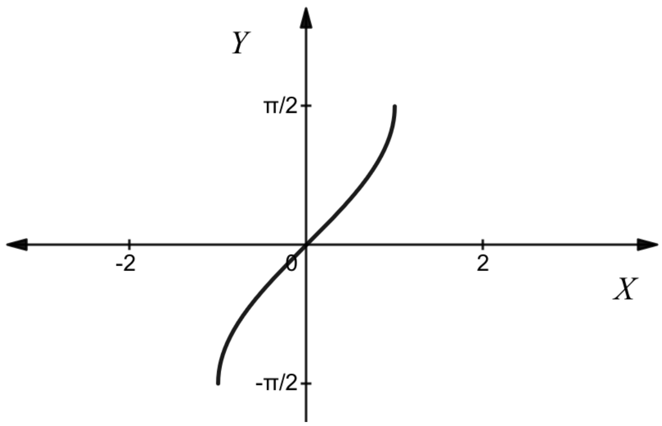
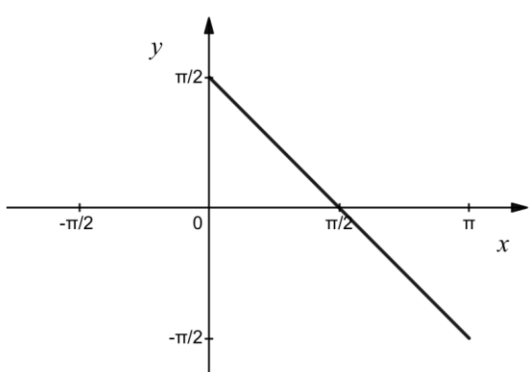
MATHEMATICS Extension 1 : Question 13

Suggested Solutions	Marks	Marker's Comments
c) $t = 2 - 2e^{-3x}$		
$\frac{dt}{dx} = 6e^{-3x}$		
$\frac{dx}{dt} = \frac{1}{6e^{-3x}}$	1	
$= \frac{e^{3x}}{6}$		
as $e^{3x} > 0$ for all $x \in R$		
then $\frac{dx}{dt} > 0$, velocity > 0	1	
$\therefore$ the particle is always moving in the positive direction.		
d) i) $P(t) = Ae^{-kt}$		
$\frac{dP}{dt} = -kAe^{-kt}$		
$= -kP(t)$	1	
$\therefore \frac{dP}{dt} \propto P$		
ii) $P(0) = Ae^{-k(0)}$		
$= A$		
$\therefore \frac{1}{2}A = Ae^{-kt}$	1	
$\frac{1}{2} = e^{-kt}$		
$-kt = \ln(\frac{1}{2})$		
$t = \frac{-\ln(\frac{1}{2})}{k}$	1	
$= \frac{\ln 2}{k}$		

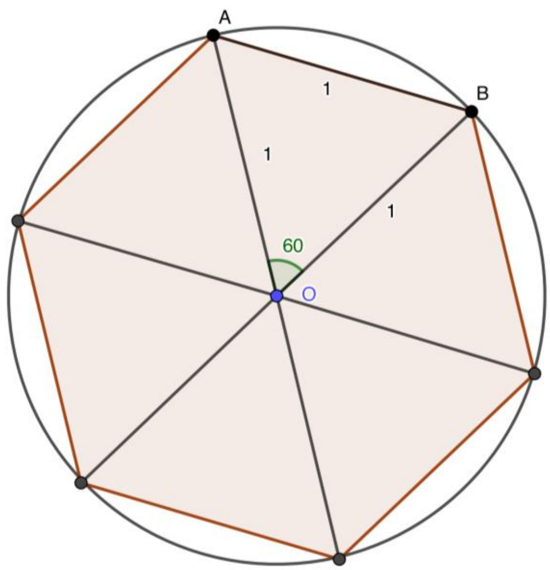
MATHEMATICS Extension 1 : Question……..

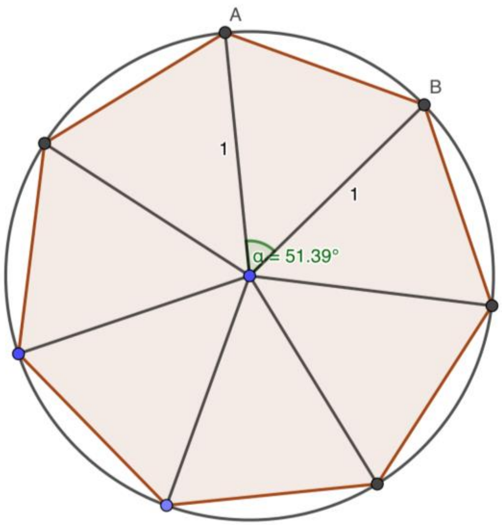
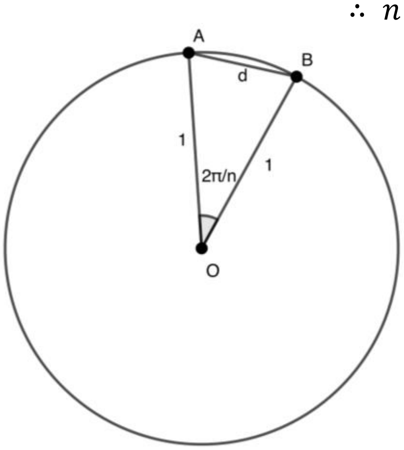
Suggested Solutions	Marks	Marker's Comments
iii) $D(t) = A - P(t)$	1	
iv) $P(t) = Ae^{-kt}$		
$P(t) = \left(D(t) + P(t)\right)e^{-kt}$	1	getting rid of A
$\dfrac{P(t)}{D(t)+P(t)} = e^{-kt}$		
$-kt = \ln\left(\dfrac{P(t)}{D(t)+P(t)}\right)$		
$t = \dfrac{-\ln\left(\frac{P(t)}{D(t)+P(t)}\right)}{k}$		
$t = \dfrac{\ln\left(\frac{D(t)+P(t)}{P(t)}\right)}{k}$	1	
$\therefore t = \frac{1}{k}\ln\left(\frac{D(t)}{P(t)} + 1\right)$		
e)  selecting any 3 points: $^{16}C_3$	1	all possible solutions
 selecting any 3 points horizontally, vertically or diagonally: $(4+4+2)\,^4C_3$		
 selecting any 3 points out of 3 diagonal points $4 \times {^3C_3}$		
$\therefore$ total $= {^{16}C_3} - (4+4+2)\,^4C_3 - 4 \times {^3C_3}$		
$= 516$	1	

MATHEMATICS EXTENSION 1 : Question 14

Suggested Solutions	Marks	Marker's Comments
a) (i) $y = \frac{\pi}{2} - \cos^{-1} x$	1 1	Shape Endpoints $\left(1, \frac{\pi}{2}\right), \left(-1, -\frac{\pi}{2}\right)$ Notes: • Graph does not have a horizontal tangent at (0,0) • Graph is not circular
		
(ii) From (i) $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$ $\therefore \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $x = \cos^{-1} t, \quad y = \sin^{-1} t$ $\therefore x + y = \cos^{-1} t + \sin^{-1} t = \frac{\pi}{2}$ $\therefore y = \frac{\pi}{2} - x$	1	Cartesian equation $y = \sin^{-1}(\cos x)$ also accepted
Domain: $0 \leq x \leq \pi$ Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	1 1	Domain and range must be identified with 'D' and 'R'
	1	Graph correct from Domain and Range

MATHEMATICS Extension 1: Question 14		
Suggested Solutions	**Marks**	**Marker's Comments**
b)		
$\tan^{-1}\left(\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$		
$= \tan^{-1}\left(\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$	1	Multiplying by conjugate
$= \tan^{-1}\left(\frac{1+\sin x+1-\sin x-2\sqrt{1+\sin x}\times\sqrt{1-\sin x}}{1+\sin x-(1-\sin x)}\right)$		
$= \tan^{-1}\left(\frac{2-2\sqrt{1-\sin^2 x}}{2\sin x}\right)$		
$== \tan^{-1}\left(\frac{2-2\sqrt{\cos^2 x}}{2\sin x}\right)$		
$= \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) \quad since\ \ x \in \left(0, \frac{\pi}{4}\right), \quad \cos x > 0 \quad (*)$	1	Simplifying
$= \tan^{-1}\left(\frac{1-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}\right) where\ t = tan\frac{x}{2}$		
$= \tan^{-1}\left(\frac{1+t^2-1+t^2}{2t}\right)$	1	Substituting $t = \tan\frac{x}{2}$
$= \tan^{-1}\left(\frac{2t^2}{2t}\right)$		
$= \tan^{-1}(t)$		
$= \tan^{-1}\left(tan\frac{x}{2}\right)$		
$= \frac{x}{2}, \quad x \in \left(0, \frac{\pi}{4}\right)$	1	Reason for removing square root at (*) and for $\tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2}$

Method 2: $$\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$$ $$=\tan^{-1}\left(\frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$ $$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$ $$=\tan^{-1}\left(\tan\frac{x}{2}\right)$$ $$=\frac{x}{2}\quad \text{since } x\in\left(0,\frac{\pi}{4}\right)$$		
c) **Method 1:** Place 6 points evenly spaced on the circumference of a circle radius 1m as in diagram. 6 equilateral triangles are formed as angle at centre is 60^o.	1	Diagram dividing circle into 6 congruent sectors by evenly spacing points around circle
The (straight line) distance between any two of the 6 points is 1m.	1	Explaining why 6 points are 1m apart
If a 7^{th} point is added on the circumference it will lie between 2 of the 6 points and its (straight line) distance from the nearest point will be <1m.  	1	Adding 7^{th} point and explanation **Note:** • Must state points are evenly spaced • Must use straight line distance – arc length only is not sufficient

Method 2: Place 7 points evenly spaced on the circumference of a circle, dividing the circle into 7 congruent sectors each with a centre angle of $\frac{2\pi}{7}$. Distance between two points: $$d^2 = 1^2 + 1^2 - \left(2 \times 1 \times 1 \times \cos\frac{2\pi}{7}\right)$$ $$d \approx 0.87 < 1$$  **Method 3:** Dividing circle into n congruent sectors by placing n points on the circumference of the circle, with angle at centre of $\frac{2\pi}{n}$. Find number of points to give a distance between points of < 1m $$AB^2 = OA^2 + OB^2 - 2OA \times OB \times \cos\frac{2\pi}{n}$$ $$= 2 - 2\cos\frac{2\pi}{n}$$ $$= 2 - 2\cos\frac{2\pi}{n} < 1$$ $$2\cos\frac{2\pi}{n} > \frac{1}{2} = \cos\frac{\pi}{3}$$ $$\therefore\ n > 6$$  		

MATHEMATICS Extension 1: Question 14		
Suggested Solutions	**Marks**	**Marker's Comments**
d) Method 1: $\frac{\text{Number of ways of arranging FOOLISH}}{\text{Number of ways of arranging LOSO}} = \frac{7!}{2!} \div \frac{4!}{2!} = 210 \text{ ways}$	1	
Arranging FOOLISH in $\frac{7!}{2!}$ ways	1	
Arranging LOSO in $\frac{4!}{2!}$ ways	1	
Method 2:		
_ L _ O _ S _ O _		
5 spaces for 3 remaining letters	1	
Placing one letter creates another space	1	
Eg. _ L _ F _ O _ S _ O _		
$\therefore 5 \times 6 \times 7 = 210 \text{ ways}$	1	
Method 3: LOSO can be represented by 4 stars ****		
**** I F H Arrange stars and 3 letters		
$\therefore \frac{7!}{4!} = 210 \text{ ways}$		
Method 4: Taking cases Case 1: LOSO = 60 ways Arrange 4 other letters with LOSO in $\frac{5!}{2}$ ways Case 2: O_ _ O = 72 ways Choose one of 2 spaces for S in 2 ways Choose one of I, F, H to go in remaining space between Os in 3 ways Arrange 3 remaining letters and O_ S O in $\frac{4!}{2}$ ways with L before OSO Case 3: O _ _ _ O = 54 ways Choose one of 3 spaces for S in 3 ways Choose 2 of I,F,H and arrange in remaining spaces between Os in 3×2 ways Arranging 2 remaining letters and O _ _ S O in $\frac{3!}{2}$ ways with L before OSO Case 4: L O _ _ _ _ O = 24 ways Arrange I, F, H and S between Os in 4! ways Total number of arrangements = 60 + 72 + 54 + 24 = 210 ways		1 mark for one correct case 2 marks for 2 correct cases 3 marks for 4 correct cases

Solution	Mark	Comment
e) RTP $\dfrac{(n^2)!}{(n!)^n} \in \mathbb{N} \quad \forall n \in \mathbb{N}$. Note: $\dfrac{(n^2)!}{(n!)^n} = \dfrac{(n^2)!}{\underbrace{n! \times n! \times \cdots \times n!}_{n \text{ times}}}$		
Denominator suggests situation: n categories, each w/ n indistinguishable objects in those categories. 	1	Defining a scenario for denominator
Form an arrangement of those objects. E.g. $\underbrace{*_1 *_2 *_2 *_5 *_5 *_5 \cdots *_n *_1 *_n}_{n^2 \text{ letters}}$ The # of words formed from n^2 letters w/ n copies of $*_1$, n " " $*_2$, ⋮ n " " $*_n$ is $\dfrac{(n^2)!}{n! \cdot n! \cdot \cdots \cdot n!}$ $= \dfrac{(n^2)!}{(n!)^n}$	1	Defining a scenario for numerator
and this must be a positive integer for $n \in \mathbb{N}$. $\therefore \forall n \in \mathbb{N}, \quad \dfrac{(n^2)!}{(n!)^n} \in \mathbb{N}$ //		Must state why result is in N for second mark