Year 11 Extension 1 online Assessment Task 3

Questions 41-44 (20 marks)
Please start each question on a new page.

41.

A pot of soup is to be refrigerated overnight. It needs to be cooled from its temperature of 100°C before it can be placed in the fridge.

The pot of soup is cooled in a sink full of cold water kept at a temperature of 5°C. The temperature of the pot of soup, $T^\circ\text{C}$, after t minutes in the sink satisfies the equation

$$\frac{dT}{dt} = -k(T-5).$$

(i) Show that $T = 5 + 95e^{-kt}$ satisfies both this equation and the initial condition. **1**

(ii) After 10 minutes in the sink, the temperature of the pot of soup reduces from 100°C to 60°C. It must be cooled to a temperature of 20°C before it can be placed in the fridge.

How long (to the nearest minute) must the pot of soup remain in the sink full of cold water? **3**

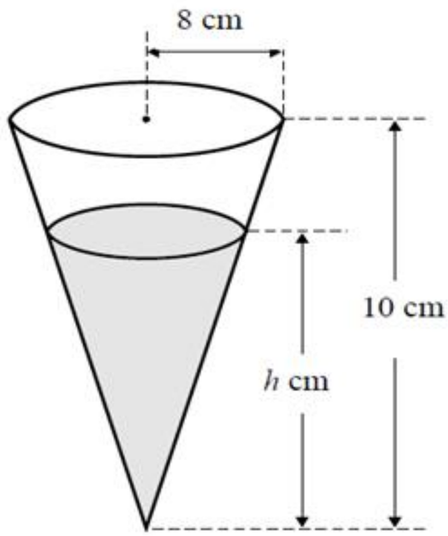
42. a. Find the exact term independent of x in the expansion of

$$(1+2x)^2\left(2x+\frac{1}{x}\right)^{12}$$ **2**

b. Use the binomial expansion of $(1+x)^n$, to prove that

$$\binom{n}{1} + 4\binom{n}{2} + 12\binom{n}{3} + \cdots + r\binom{n}{r}2^{r-1} + \cdots + n\binom{n}{n}2^{n-1} = n.3^{n-1}$$ 2

43.



The figure above shows an inverted conical cup with base radius 8 cm and height 10 cm.

Some water is poured into the cup at a constant rate of $\frac{2\pi}{5}$ cm^3 per minute.

Let the depth of the water be h cm at time t minutes.

4

Find the rate of change in the area of the water surface when $h = 4$

44. Given that $m = a(\cos x)^3 + 3a\cos x\,(\sin x)^2$

and $n = a(\sin x)^3 + 3a\sin x\,(\cos x)^2$

Prove that $(m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$ 4

Q41

i) $T = 5 + 95e^{-kt}$ (*)

$\therefore T - 5 = 95e^{-kt}$ (1)

$\frac{dT}{dt} = -k \times 95e^{-kt}$

$\frac{dT}{dt} = -k(T-5)$, from (1)

so (*) satisfies the equation

$t = 0 \rightarrow T = 5 + 95e^{-k \times 0}$

$= 5 + 95$

$\therefore T = 100$

so (*) satisfies the initial condition.

1 need BOTH

ii) when $t = 10$, $T = 60$

$\therefore 60 = 5 + 95e^{-k \times 10}$

$55 = 95e^{-10k}$

$\frac{11}{19} = e^{-10k}$

$-10k = \ln\left(\frac{11}{19}\right)$

$k = \frac{1}{10}\ln\left(\frac{19}{11}\right)$ 1

we want t when $T = 20$

$20 = 5 + 95e^{-kt}$

$\frac{3}{19} = e^{-kt}$

$-kt = \ln\left(\frac{3}{19}\right)$

$t = \frac{1}{k}\ln\left(\frac{19}{3}\right)$

$= \frac{10\ln\left(\frac{19}{3}\right)}{\ln\left(\frac{19}{11}\right)}$ 1

$t = 33.77 \rightarrow t = 34$ mins.

$\therefore$ must remain for a total of 34 mins

① final

OR

must remain for an additional 24 mins.

Question 42

Tuesday, 12 October 2021 12:39 pm

Find the exact term independent of x in the expansion of

$$(1+2x)^2\left(2x+\frac{1}{x}\right)^{12} \qquad 2$$

a) $(1+2x)^2\left(2x+\frac{1}{x}\right)^{12}$

$= (1+4x+4x^2)\left(2x+\frac{1}{x}\right)^{12}$

$= (1+4x+4x^2)\sum_{r=0}^{12} {}^{12}C_r\,(2x)^{12-r}(x^{-1})^r$

$= (1+4x+4x^2)\sum_{r=0}^{12} {}^{12}C_r\,2^{12-r}\,x^{12-2r}$

For term independent of x:

either $12-2r=0$ or $12-2r=-2$

$[12-2r \neq 1$ since $r \in \mathbb{Z}^+]$

$\therefore r=6$ or $r=7$

$\therefore$ term independent of x

$= 1 \times {}^{12}C_6 \cdot 2^6 + 4 \times {}^{12}C_7\, 2^5$

$= 160\,512$

1 mark for $r=6$ and $r=7$
or for $r=6$ and term

1 mark for total value of term.

Use the binomial expansion of $(1+x)^n$, to prove that

$$\binom{n}{1}+4\binom{n}{2}+12\binom{n}{3}+\cdots+r\binom{n}{r}2^{r-1}+\cdots+n\binom{n}{n}2^{n-1}=n.3^{n-1} \qquad 2$$

b) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n$

Differentiating both sides:

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + 3\,{}^nC_3 x^2 + \cdots + r\,{}^nC_r x^{r-1} + \cdots + n\,{}^nC_n x^{n-1}$

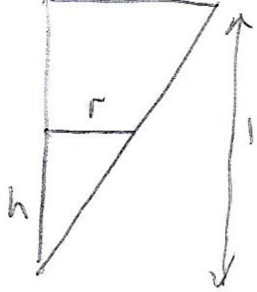
Let $x=2$:

$n\,3^{n-1} = {}^nC_1 + 2\,{}^nC_2 \times 2 + 3\,{}^nC_3 \times 2^2 + \cdots + r\,{}^nC_r\,2^{r-1} + \cdots + n\,{}^nC_n\,2^{n-1}$

$\therefore {}^nC_1 + 4\,{}^nC_2 + 12\,{}^nC_3 + \cdots + r\,{}^nC_r\,2^{r-1} + \cdots + n\,{}^nC_n\,2^{n-1} = n\,3^{n-1}$

1 mark for differentiating both sides with explanation

1 mark for substituting $x=2$ with explanation and showing final statement.



$$\frac{r}{h} = \frac{8}{10} \therefore r = \frac{4h}{5} \quad \checkmark$$

$$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt} \therefore \frac{dh}{dt} = \frac{dv/dt}{dv/dh}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{16}{75}\pi h^3 \quad \checkmark$$

$$\frac{dv}{dh} = \frac{16}{25}\pi h^2 \text{ and } \frac{dv}{dt} = \frac{2\pi}{5} \text{ (given)}$$

$$\frac{dh}{dt} = \frac{2\pi/5}{\frac{16}{25}\pi h^2} = \frac{5}{8h^2} \quad \checkmark$$

$$A = \pi r^2 = \frac{16}{25}\pi h^2$$

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$$

$$= \frac{32}{25}\pi h \times \frac{5}{8h^2} = \frac{4\pi}{5h} \quad \checkmark$$

For $h = 4$, $\frac{dA}{dt} = \frac{\pi}{5}$.

$$h = \sqrt{\frac{25}{16\pi}A} = \frac{5}{4}\sqrt{\frac{A}{\pi}}$$

$$V = \frac{1}{3}Ah = \frac{1}{3}A\,\frac{5}{4}\left(\frac{A}{\pi}\right)^{1/2} = \frac{5}{12\sqrt{\pi}}A^{3/2}$$

$$\frac{dv}{dA} = \frac{5}{8\sqrt{\pi}}A^{1/2} = \frac{5}{8\sqrt{\pi}} \times \frac{4}{5}\sqrt{\pi}h = \frac{h}{2}$$

$$\frac{dA}{dt} = \frac{dA}{dV} \times \frac{dv}{dt} = \frac{2}{h} \times \frac{2\pi}{5} = \frac{4\pi}{5h}$$

Q44.

Given that $m = a(\cos x)^3 + 3a\cos x\,(\sin x)^2$ and $n = a(\sin x)^3 + 3a\sin x\,(\cos x)^2$. Prove that $(m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$

$$\begin{aligned} m + n &= a\cos^3 x + 3a\cos^2 x\sin x + 3a\cos x\sin^2 x + a\sin^3 x \\ &= a(\cos^3 x + 3\cos^2 x\sin x + 3\cos x\sin^2 x + \sin^3 x) \\ &= a(\cos x + \sin x)^3 \end{aligned}$$

1 mark for simplifying $m + n$

$$\begin{aligned} (m+n)^{\frac{1}{3}} &= a^{\frac{1}{3}}(\cos x + \sin x) \\ (m+n)^{\frac{2}{3}} &= a^{\frac{2}{3}}(\cos x + \sin x)^2 \\ &= a^{\frac{2}{3}}(\cos^2 x + \sin^2 x + 2\cos x\sin x) \\ &= a^{\frac{2}{3}} + 2a^{\frac{2}{3}}\cos x\sin x \end{aligned}$$

1 mark for simplifying $(m+n)^{\frac{2}{3}}$

$$\begin{aligned} m - n &= a\cos^3 x - 3a\cos^2 x\sin x + 3a\cos x\sin^2 x - a\sin^3 x \\ &= a(\cos^3 x - 3\cos^2 x\sin x + 3\cos x\sin^2 x - \sin^3 x) \\ &= a(\cos x - \sin x)^3 \end{aligned}$$

$$\begin{aligned} (m-n)^{\frac{1}{3}} &= a^{\frac{1}{3}}(\cos x - \sin x) \\ (m-n)^{\frac{2}{3}} &= a^{\frac{2}{3}}(\cos x - \sin x)^2 \\ &= a^{\frac{2}{3}}(\cos^2 x + \sin^2 x - 2\cos x\sin x) \\ &= a^{\frac{2}{3}} - 2a^{\frac{2}{3}}\cos x\sin x \end{aligned}$$

1 mark for simplifying $(m-n)^{\frac{2}{3}}$

$$\begin{aligned} \therefore (m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}} &= \left(a^{\frac{2}{3}} + 2a^{\frac{2}{3}}\cos x\sin x\right) + \left(a^{\frac{2}{3}} - 2a^{\frac{2}{3}}\cos x\sin x\right) \\ &= 2a^{\frac{2}{3}} \end{aligned}$$

1 mark for finishing the proof