Student Name:_____

Teacher Name:_____

2023 **Glenwood High School**
Year 11 – Yearly Examination

Mathematics Extension 1

General Instructions

* **Reading Time – 5 minutes**
* **Working time – 1 hour 30 minutes**
* Write using black pen
* NESA approved calculators may be used
* A reference sheet is provided
* Show relevant mathematical reasoning and/or calculations

Total marks: 52

Section I – 7 marks (pages 2 – 5)

* Attempt Questions 1 - 7
* Allow about 12 minutes for this section

Section II – 45 marks (pages 8-12)

* Attempt Questions 8 – 10
* Allow about 1 hour 18 minutes for this section

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Section I

7 marks
Attempt questions 1–7
Allow about 12 minutes for this question

Use the provided answer sheet for Questions 1–7.

1 The number, N, of bacteria in a colony after t minutes is given by $N = 5000e^{kt}$. After 5 minutes the population of the colony doubled. What is the value of k to 4 significant figures.

A. 0.3216

B. 0.1386

C. 0.138

D. 0.8047

2 Considering the expansion of $(1 + x)^7$, which statement is **not** true.

A. One of the terms is $21\ x^5$.

B. The constant term is 1.

C. The general term is of the form ${}^7\mathbf{C}_k\ .\ x^k$.

D. There are seven terms altogether.

3 What does the expression $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x}$ simplify to?

A. $\cot x$

B. $-\tan x$

C. $\tan x$

D. $-\cot x$

4 On the afternoon before a street party, Angela places a few small gifts in each of the 16 letterboxes along her block. She has 90 small gifts which she distributes by placing one in each box, then going back and placing another in each box, until she has run out of gifts. Which statement could be made based on the pigeonhole principle?

A. All letterboxes will have exactly six gifts in them

B. All letterboxes will have at least six gifts in them

C. No letterboxes will have exactly six gifts in them

D. There is at least one letterboxes with six gifts in it

5 A shop has 12 employees consisting of 5 men and 7 women. How many groups of 5 employees can be chosen which contain exactly 3 females?

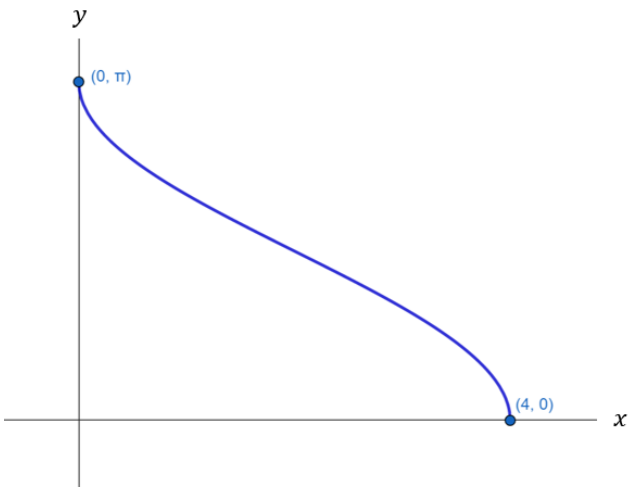
A. 792

B. 210

C. 350

D 45

6



The equation of the graph shown above could be

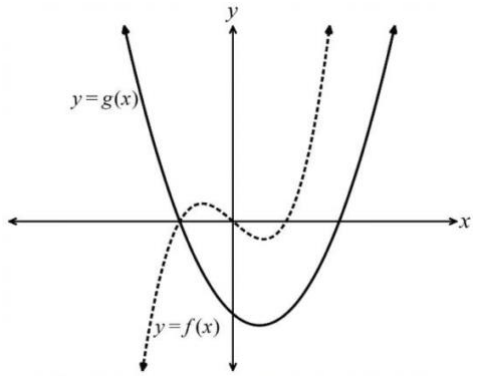
A. $y = \cos^{-1}\left(\frac{x}{4}\right) + \frac{\pi}{2}$

B. $y = \cos^{-1}\left(\frac{x}{2}\right)$

C. $y = \cos^{-1}\left(\frac{x}{2}\right) - 1$

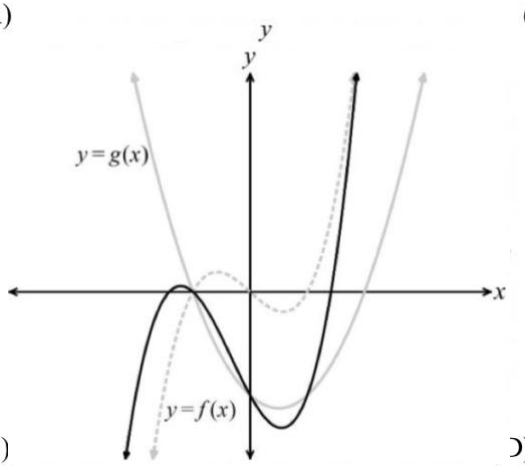
D. $y = \cos^{-1}\left(\frac{x}{2} - 1\right)$

7 The graph of $y = f(x)$ and $y = g(x)$ are shown on the set of axes below.

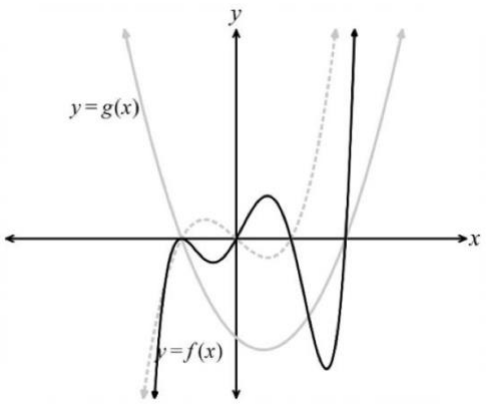


Which diagram below shows the graph of $y = f(x) + g(x)$

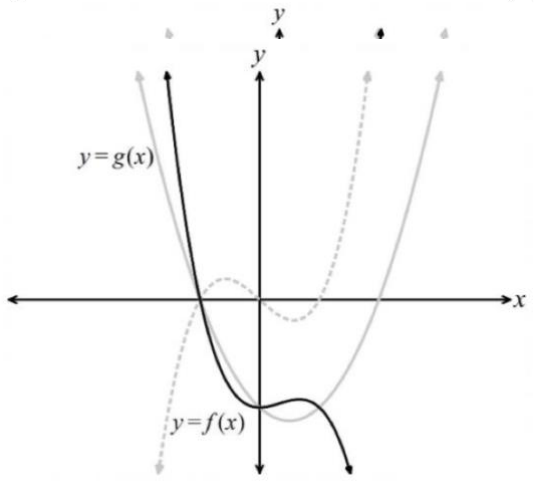
(A)



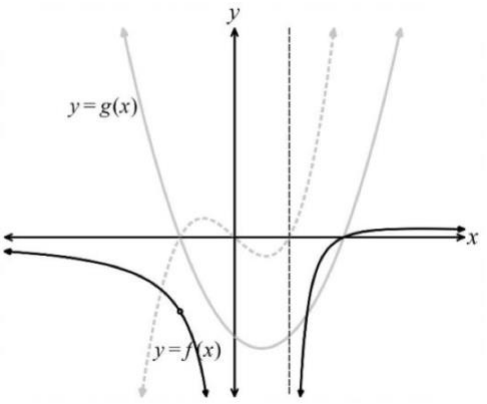
(B)



(C)



(D)



End of Section I

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Student Name: _____

2023 **YEAR 11 YEARLY EXAMINATION**

Mathematics Extension 1

Section II

45 marks
Attempt Questions 8–10
Allow about 1 hours and 18 minutes for this section

Instructions

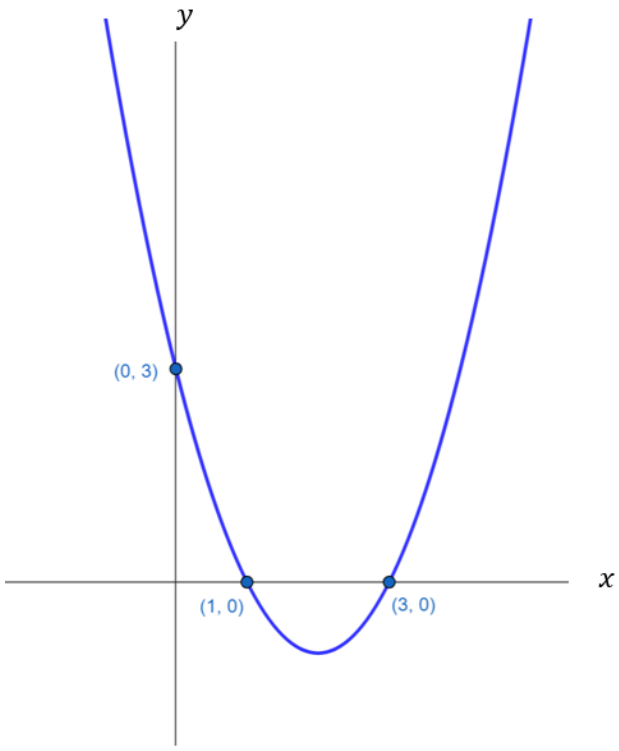
- Answer each question in a SEPARATE writing booklet.
- Your responses should include relevant mathematical reasoning and/or calculations.

Please turn over

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Solve: $\frac{x}{x-1} \geq 2$ **3**

(b) The graph shows a function, $f(x)$, with its x and y-intercepts labelled.



In your writing booklet, draw a half-page sketch of the graph of $y = \frac{1}{f(x)}$. Label all key features including the y-intercept. **2**

(c) In how many ways can the letters of the word 'COMMITTEE', taken all at a time , be arranged if:

(i) there is no restriction? **1**

(ii) the two E's stay together? **1**

Question 8 continues on page 9

(d) Let $P(x) = (x-2)(x+1)A(x) + a(x+1) + b$ where *A(x)* is a polynomial. a and b are real numbers.
When $P(x)$is divided by $(x+1)$, the remainder is -10.
When $P(x)$ is divided by $(x-2)$, the remainder is 2.

(i) Find the values of a and b. **2**

(ii) What is the remainder when $P(x)$ is divided by $(x-2)(x+1)$? **1**

(e) Consider the function

$$f(x) = 1 + \frac{3}{x-4}$$

for $x > 4$

(i) Find $f^{-1}(x)$ **2**

(ii) State the domain of $f^{-1}(x)$ **1**

(f) Find the Cartesian equation of the curve represented by the parametric equations below. **2**

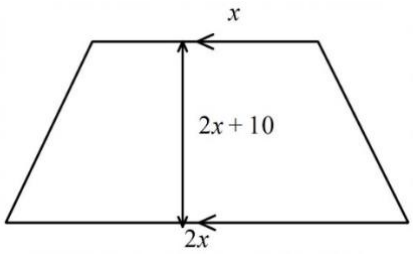
$$x = 5 - 2t$$
$$y = t^2 - t$$

End of Question 8

Question 9 (16 marks) Use a SEPARATE writing booklet.

(a) What is the exact value of $\tan\left(sin^{-1}\left(-\frac{2}{3}\right)\right)$? **2**

(b) The area of the trapezium below is changing as the value of x changes over time. **2**



Find an expression for the area of the trapezium in terms of x, and hence find $\frac{dA}{dx}$.

(c) Given that $t = \tan\frac{A}{2}$, prove: **2**

$$\frac{1 + cosA}{secA - tanA} = \frac{2(1+t)}{(1-t)(1+t^2)}$$

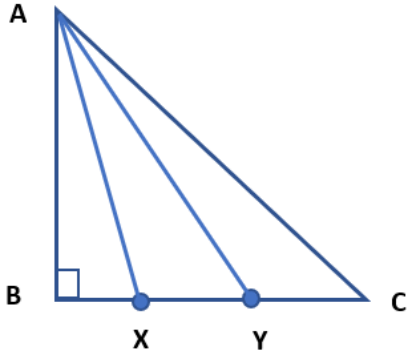
(d) Given that for $P(x) = x^4 - 7x^3 + 9x^2 + 27x - 54$, the derivative function $P'(x)$ has a root of multiplicity 2 at $x = 3$, factorise $P(x)$ into linear factors **2**

Question 9 continues on page 11

Question (9 continued)

(e) In the isosceles triangle below, $AB = BC$, $\angle ABC = 90°$ and points X and Y are placed such that $BX = XY = YC = 1$ **4**

Let $\angle BAX = \alpha$, $\angle XAY = \beta$ and $\angle YAC = \gamma$



Find the exact values of $\tan\alpha$, $\tan\beta$ and $\tan\gamma$.

(f) A particle moves along the x-axis so that its displacement in metres from the origin at a time t seconds is given by the equation $x = e^{t^2-4t}$.

(i) What is the initial displacement and velocity of the particle? **2**

(ii) Find an exact value for the acceleration of the particle, when it is at rest. **2**

End of Question 9

Please turn over for Question 10

Question 10 (14 marks) Use a SEPARATE writing booklet.

(a) Find the term independent of x in the expansion of $\left(3x + \frac{1}{x^2}\right)^9$. **3**

(b) Use the expansion of $cos\,30°\,sin\,15°$ to show that **2**

$$sin15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(c) By expanding both sides of the identity $(1+x)^{11} = (1+x)^8(1+x)^3$, **3**
Prove that : $\binom{11}{3} = \binom{8}{0}\binom{3}{3} + \binom{8}{1}\binom{3}{2} + \binom{8}{2}\binom{3}{1} + \binom{8}{3}\binom{3}{0}$

(d) A scientist has a 100 gram sample of a radioactive substance. The mass of the substance decreases due to radioactive decay.

After 5 days, the sample's mass decreases to 80 grams.

After 5 **more** days, the sample's mass decreases to 65 grams.

Assume that the mass m grams of the substance after t days is modelled by the differential equation

$$\frac{dm}{dt} = -r(m - A)$$

where r and A are constants.

(i) Show that the equation $m = A + Be^{-rt}$, where B is some constant, satisfies the above differential equation. **1**

(ii) Find the exact values of r, A and B. **4**

(iii) According to safety guidelines, the environment around the radioactive substance will only be safe when the mass of the substance has decreased to 15 grams or less. **1**

Explain, with mathematical reasoning, whether the environment will ever be safe.

END OF PAPER

8) a) Solve: $\frac{x}{x-1} \geq 2$ $\quad x-1 \neq 0$, $x \neq 1$

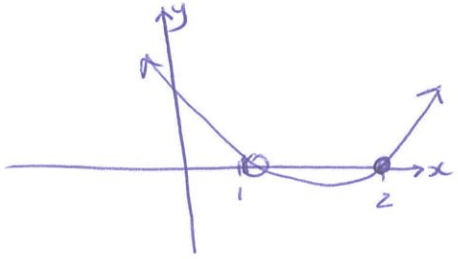
$x(x-1)^2$

$x(x-1) \geq 2(x-1)^2$

$2(x-1)^2 - x(x-1) \leq 0$

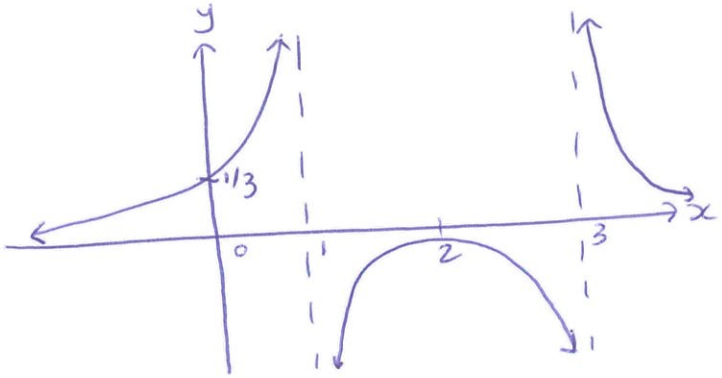
$(x-1)(2(x-1)-x) \leq 0$

$(x-1)(x-2) \leq 0$



$1 < x \leq 2$

b)



c) (i) $\frac{9!}{2!2!2!} = 45360$

(ii) $\frac{8! \times 2}{2! \times 2! \times 2!} = 10\,080$

d) $P(x) = (x-2)(x+1)A(x) + a(x+1) + b$

$P(-1) = -10$

$P(2) = 2$

(i) $-10 = (-1-2)(-1+1)A(-1) + a(0) + b$

$-10 = b$

$2 = (2-2)(2+1)A(2) + a(2+1) - 10$

$2 = 3a - 10$

$12 = 3a$

$4 = a$

Y11 Ext 1 Yearly Exam 2023

(1) $t=0 \quad N = 5000e^{0}$

$= 5000$

$t=5 \quad N = 10000$

$10000 = 5000e^{5k}$

$\ln 2 = 5k$

$\frac{\ln 2}{5} = k$

$0.1386 = k$ B

(2) D (There are 8 terms, not 7)

(3) $\frac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x}$

$= \frac{\cos(4x-x) - \cos(4x+x)}{\sin(4x-x) - \sin(4x+x)}$

$= \frac{\cos 4x\cos x + \sin 4x \sin x - \cos 4x \cos x + \sin 4x \sin x}{\sin 4x \cos x - \cos 4x \sin x + \sin 4x \cos x + \cos 4x \sin x}$

$= \frac{2\sin 4x \sin x}{2 \sin 4x \cos x}$

$= \frac{\sin x}{\cos x}$

$= \tan x$ C.

(4) $\frac{90}{16} = 5.625$ D

(5) ${}^{7}C_{3} \times {}^{5}C_{2} = 350$ C

(6) $0 \leq x \leq 4$

$0 \leq \frac{x}{2} \leq 2$

$-1 \leq \frac{x}{2} - 1 \leq 1$ D

(7) A

8 d ii) This equation is written in the form:

$$P(x) = Q(x)(x-a) + R$$

R: $a(x+1)+b$

$= 4(x+1) - 10$

$= \underline{4x-6}$.

e) $f(x) = 1 + \frac{3}{x-4}$ $\quad x > 4$ so $y > 1$

(i) $x = 1 + \frac{3}{y-4}$

$x - 1 = \frac{3}{y-4}$

$y - 4 = \frac{3}{x-1}$

$f^{-1}(x) = y = \frac{3}{x-1} + 4$

(ii) domain of $f^{-1}(x)$ is range of $f(x)$ so $x > 1$ or $(1, +\infty)$

f) $x = 5 - 2t$ ① $\longrightarrow$ $t = \frac{5-x}{2}$

$y = t^2 - t$ ②

sub ① into ②

$y = \left(\frac{5-x}{2}\right)^2 - \frac{5-x}{2}$

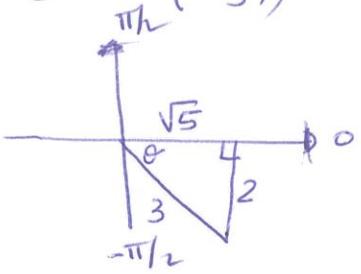
$= \frac{25 - 10x + x^2}{4} - \frac{5-x}{2}$

$= \frac{x^2 - 10x + 25 - 10 + 2x}{4}$

$= \frac{x^2 - 8x + 15}{4}$

Q9

a) $\tan\left(\sin^{-1}\left(-\frac{2}{3}\right)\right)$ ← (in Q4 as $\sin\theta$ is -ve)



$\tan\theta = -\frac{2}{\sqrt{5}}$ (tan is -ve in Q4)

b) $A = \left(\frac{2x+10}{2}\right)(2x+x)$

$= (x+5)(3x)$

$= 3x^2+15x$

$\frac{dA}{dx} = 6x+15$

c) $t = \tan\frac{A}{2}$

LHS $= \frac{1+\cos A}{\sec A - \tan A}$

$= \left(1 + \frac{1-t^2}{1+t^2}\right) \div \left(\frac{1+t^2}{1-t^2} - \frac{2t}{1-t^2}\right)$

$= \frac{1+t^2+1-t^2}{1+t^2} \times \frac{1-t^2}{t^2-2t+1}$

$= \frac{2(1-t^2)}{(1+t^2)(t-1)^2}$

$= \frac{2(1+t)(1-t)}{(1+t^2)(t-1)(t-1)}$

$= \frac{-2(1+t)(t-1)}{(1+t^2)(t-1)(t-1)}$

$= \frac{2(1+t)}{(1+t^2)(1-t)}$

$=$ RHS

9

d) $P(x) = x^4 - 7x^3 + 9x^2 + 27x - 54$

$P'(x) = 4x^3 - 21x^2 + 18x + 27$

$P''(x) = 12x^2 - 42x + 18$ ← not needed.

$P(x)$ will have a root of multiplicity 3 at $x = 3$

so $P(x) = (x-3)^3 . Q(x)$ and $Q(x) = (x - a)$ (coeff of x^4 in $P(x) = 1$

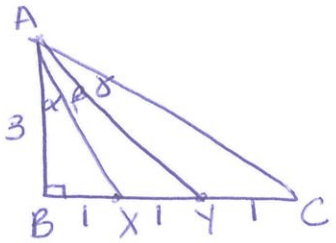
Constant term of $P(x) = -54$ comes from $(-3)^3 . a$

$= -27a = 54$

$a = -2$

$\therefore P(x) = (x-3)^3(x+2)$

e)



$\tan\alpha = \frac{O}{A}$

$= \frac{1}{3}$

$\tan\beta = ?$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\frac{2}{3} = \frac{\frac{1}{3} + \tan\beta}{1 - \frac{1}{3}\tan\beta}$ $\left(\tan(\alpha+\beta) = \frac{O}{A} = \frac{2}{3}\right)$

$= \frac{1+3\tan\beta}{3} \div \frac{3 - \tan\beta}{3}$

$\frac{2}{3} = \frac{1+3\tan\beta}{3-\tan\beta}$

$\frac{6 - 2\tan\beta}{3} = 3 + 9\tan\beta$

$3 - 11\tan\beta$

$\frac{3}{11} = \tan\beta$

$\tan(\alpha+\beta+\gamma) = \frac{O}{A} = 1 = \frac{\tan(\alpha+\beta) + \tan\gamma}{1 - \tan(\alpha+\beta)\tan\gamma}$

$1 = \left(\frac{2}{3} + \tan\gamma\right) \div \left(1 - \frac{2\tan\gamma}{3}\right)$

$= \frac{2 - 3\tan\gamma}{3} \div \left(\frac{3 - 2\tan\gamma}{3}\right)$

9.
e) cont'd.

$$1 = \frac{2+3\tan\gamma}{3-2\tan\gamma}$$

$$3-2\tan\gamma = 2+3\tan\gamma$$

$$1 = 5\tan\gamma$$

$$\frac{1}{5} = \tan\gamma$$

$\therefore \tan\alpha = \frac{1}{3} \quad \tan\beta = \frac{3}{11} \quad \tan\gamma = \frac{1}{5}$

f) $x = e^{t^2-4t}$

(i) $t=0 \quad x = e^{0-0} = 1\text{ m}$

$$\frac{dx}{dt} = (2t-4)\,e^{t^2-4t}$$

at $t=0$

$$= (2(0)-4)\,e^{0}$$

$$= -4\text{ m/s}$$

[so initially 1m to the right of the origin moving towards the origin at 4m/s].

(iii) $\frac{d^2x}{dt^2} = ?$

$u = 2t-4 \qquad v = e^{t^2-4t}$

$u' = 2 \qquad v' = (2t-4)\,e^{t^2-4t}$

$$\frac{d^2x}{dt^2} = 2e^{t^2-4t} + (2t-4)(2t-4)\,e^{t^2-4t}$$

$$= e^{t^2-4t}\left(2+(2t-4)^2\right)$$

when is particle at rest ie. $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = (2t-4)\left(e^{t^2-4t}\right)$$

at rest when $2t-4=0$ (as e^{t^2-4t} is never $=0$.)

$$t=2$$

sub $t=2$ into $\frac{d^2x}{dt^2}$

$$\frac{d^2x}{dt^2} = e^{2^2-4(2)}\left(2+(2(2)-4)^2\right)$$

$$= e^{-4}(2+0)$$

$$= \frac{2}{e^4}$$

Q.

a) $\left(3x+\frac{1}{x^2}\right)^9$

term independent of x

$T_{r+1} = {}^nC_r\, a^{n-r} b^r$

$T_{r+1} = {}^9C_r\, (3x)^{9-r} \left(x^{-2}\right)^r$

$= {}^9C_r \cdot 3^{9-r} (x)^{9-r}\, x^{-2r}$

$x^{9-r} \cdot x^{-2r} = x^0$ (for constant term)

$9-3r = 0$

$r = 3$

constant term is ${}^9C_3 \cdot 3^{9-3}$

$= \underline{61236}$.

b) $\cos A \sin B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

$\cos 30^\circ \sin 15^\circ = \frac{1}{2}\left(\sin(30^\circ+15^\circ) - \sin(30^\circ-15^\circ)\right)$

$\frac{\sqrt{3}}{2} \sin 15^\circ = \frac{1}{2}\left(\sin 45^\circ - \sin 15^\circ\right)$

$\sqrt{3} \sin 15^\circ = \frac{1}{\sqrt{2}} - \sin 15^\circ$

$\sin 15^\circ + \sqrt{3} \sin 15^\circ = \frac{1}{\sqrt{2}}$

$\sin 15^\circ (1+\sqrt{3}) = \frac{1}{\sqrt{2}}$

$\sin 15^\circ = \frac{1}{\sqrt{2}(1+\sqrt{3})} \cdot \frac{\sqrt{2}(1-\sqrt{3})}{\sqrt{2}(1-\sqrt{3})}$

$= \frac{\sqrt{2}(1-\sqrt{3})}{2(1-3)}$

$= \frac{\sqrt{2}-\sqrt{6}}{-4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

c) $(1+x)^{11} = (1+x)^8 (1+x)^3$

$^{11}C_3$ is coeff of x^3 term in expansion of $(1+x)^{11}$

Expanding $(1+x)^8 (1+x)^3$, the terms involving x^3 will be:

$^8C_3 x^3 . {}^3C_0 + {}^8C_2 x^2 {}^3C_1 x + {}^8C_1 x . {}^3C_2 x^2 + {}^8C_0 . {}^3C_3 x^3$

$\therefore$ coeff of x^3 on RHS of original equation is

$\binom{8}{0}\binom{3}{3} + \binom{8}{1}\binom{3}{2} + \binom{8}{2}\binom{3}{1} + \binom{8}{3}\binom{3}{0}$

since coeff on both sides of eq^n for x^3 should be the same:

$$\binom{11}{3} = \binom{8}{0}\binom{3}{3} + \binom{8}{1}\binom{3}{2} + \binom{8}{2}\binom{3}{1} + \binom{8}{3}\binom{3}{0}$$

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d) $m = A + Be^{-rt}$ (1)

$\frac{dm}{dt} = -rBe^{-rt}$

but since $m = A + Be^{-rt}$

$m - A = Be^{-rt}$ from eq^n (1).

so $\frac{dm}{dt} = -r(m-A)$

(ii) $m = A + Be^{-rt}$

$t = 0$ $m = 100$

$100 = Be^0 + A$

$B = 100 - A$

so $m = A + (100-A)e^{-rt}$

$t = 5$ $m = 80$

$80 = A + (100-A)e^{-5r}$ (2)

$t = 10$ $m = 65$

$65 = A + (100-A)e^{-10r}$ (3).

can solve simultaneously

(2) / (3)

$$\frac{80-A}{65-A} = \frac{(100-A)e^{-5r}}{(100-A)e^{-10r}}$$

$$\frac{80-A}{65-A} = e^{5r}$$

sub into eqn (2)

$$80 = A + (100-A)\cdot\frac{65-A}{80-A}$$

$$80-A = (100-A)\left(\frac{65-A}{80-A}\right)$$

$$(80-A)(80-A) = (100-A)(65-A)$$

$$6400 - 160A + A^2 = 6500 - 165A + A^2$$

$$2800 = -35A$$

$$80 = A$$

$$20 = B$$

$$r = ?$$

$$\frac{80-A}{65-A} = e^{5r}$$

$$100 = 5A$$

$$20 = A$$

$$80 = B$$

$$r = ?$$

$$\frac{80-A}{65-A} = e^{5r}$$

$$\frac{60}{45} = e^{5r}$$

$$\frac{1}{5}\ln\frac{4}{3} = r$$

10 d (iii) $m = 20 + 80e^{-rt}$

$t \to \infty$, $e^{-rt} \to 0$

$80e^{-rt} \to 0$ means $m \to 20$ (limiting mass)

so the mass of the radioactive substance will approach 20, never be less than 15, so environment will never be safe.

The End