FORT STREET HIGH SCHOOL

Year 11 Mathematics Extension 1

Task 3

2022

Name:

Teacher	Razzaghi	Kaur	Moon	Wilkinson
	▢	▢	▢	▢

General Instructions

- Reading time – 10 minutes
- Working time – 90 minutes
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for poorly set work

Total marks: 56

Section I – 6 marks

- Attempt Questions 1-6
- Allow about 8 minutes for this section

Section II – 50 marks

- Attempt questions 7-10
- Allow about 82 minutes for this section

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Section I – 6 marks

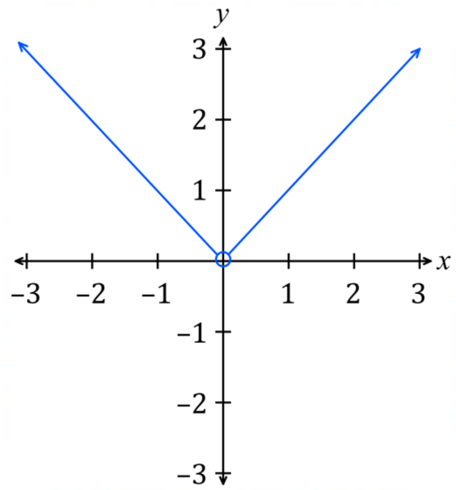
Attempt Questions 1 – 6

Allow about 8 minutes for this section.

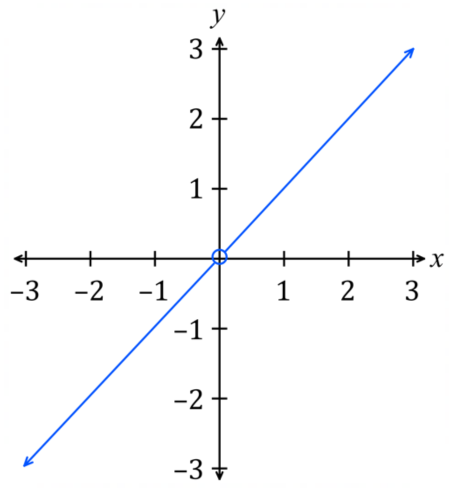
Use the multiple-choice answer sheet for Questions 1 – 6.

1. Which of the following is the graph of $f(x) = \dfrac{|x|}{x}$ for $x \neq 0$.

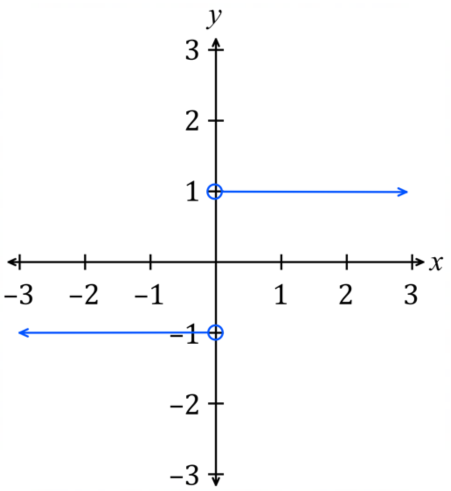
(A)



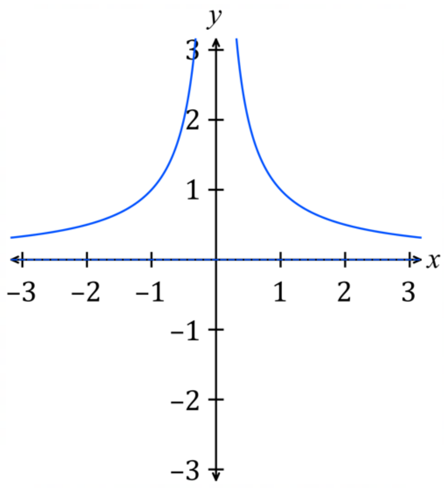
(B)



(C)



(D)



2. Which of the following is the solution to $|2x+5| \leq 17$

(A) $-11 \leq x \leq 6$

(B) $x \leq -11, x \geq 6$

(C) $-6 \leq x \leq 11$

(D) $x \leq -6, x \geq 11$

3. The polynomial $2x^3 - 9x^2 + 7x + 6$ has roots α, $-\dfrac{1}{\alpha}$ and β. The value of β is

(A) 2

(B) −2

(C) 3

(D) −3

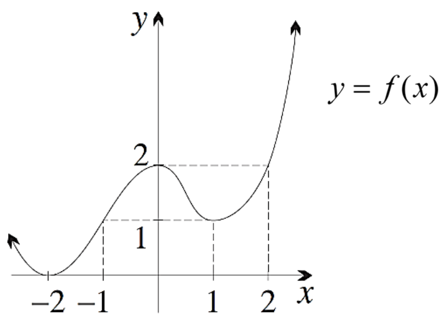
4. Given the graph of the quartic $y = f(x)$ below, which of the following restrictions on the domain would allow the function $f(x)$ to have an inverse function $y = f^{-1}(x)$?

(A) $[-2,1]$

(B) $[-2,2]$

(C) $[0,\infty)$

(D) $[-2,0]$



5. Evaluate $\sin(\alpha - \beta)$ given $\sin\alpha = \dfrac{15}{17}$ and $\sin\beta = \dfrac{3}{5}$.

(A) $\dfrac{84}{85}$

(B) $\dfrac{36}{85}$

(C) $\dfrac{13}{85}$

(D) $\dfrac{77}{85}$

6. Which of the following statements is false?

(A) The domain of $f(x) = \sin^{-1} x$ is $[-1,1]$

(B) $f(x) = \cos^{-1} x$ is an even function

(C) $f(x) = \sin^{-1} x$ is an odd function

(D) The range of $f(x) = \cos^{-1} x$ is $[0,\pi]$

Section II – 50 marks

Attempt Questions 7 – 10

Allow about 1 hour and 22 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 7 – 10, your responses should include relevant mathematical reasoning and/or calculations.

Question 7 (13 marks) – Start a new booklet

(a) Solve $\dfrac{x-4}{x-1} > 3$. **3**

(b) Let $P(x) = 3x^3 + 2x^2 - 7x + 2$.

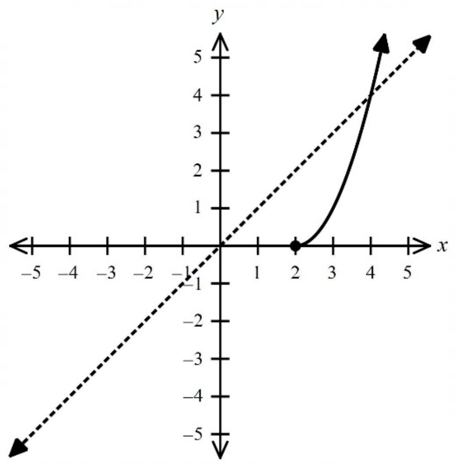
(i) Show that $(x-1)$ is a factor of $P(x)$. **1**

(ii) Hence express $P(x)$ as a product of three linear factors. **2**

(c) The graph of $f(x) = (x-2)^2;\ [2, \infty)$ is shown below, along with the line $y = x$.

(i) Write the equation of the inverse function of $f(x)$ in the form $y = f^{-1}(x)$ and state its domain. **2**

(ii) Hence sketch the inverse function, including the original function and the line $y = x$ in your sketch. **2**

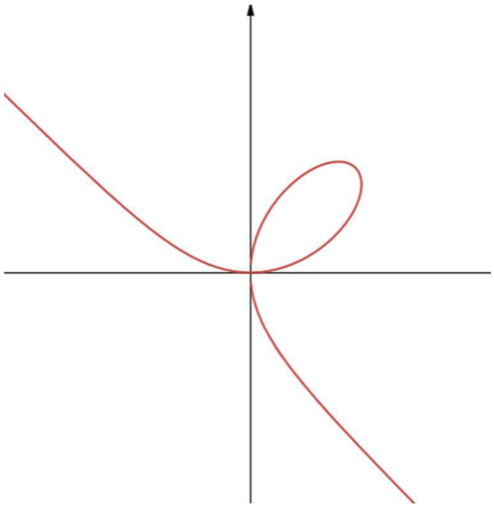


(d) Prove that $\dfrac{2\cos\theta + 1 + \cos 2\theta}{2\cos\theta - 1 - \cos 2\theta} = \cot^2 \dfrac{\theta}{2}$ **3**

Question 8 (11 marks) – Start a new booklet

(a) The graph to the right referred to as "the folium of Descartes", is for the curve defined parametrically by:

$$\begin{cases} x = \dfrac{3t}{1+t^3} \\ y = \dfrac{3t^2}{1+t^3} \end{cases}$$



(i) Find the simplest expression for $\dfrac{y}{x}$ in terms of t. **1**

(ii) Deduce that the Cartesian equation of this curve is: $x^3 + y^3 = 3xy$. **2**

(b) Consider the function $f(x) = 2\sin^{-1}(x-2)$

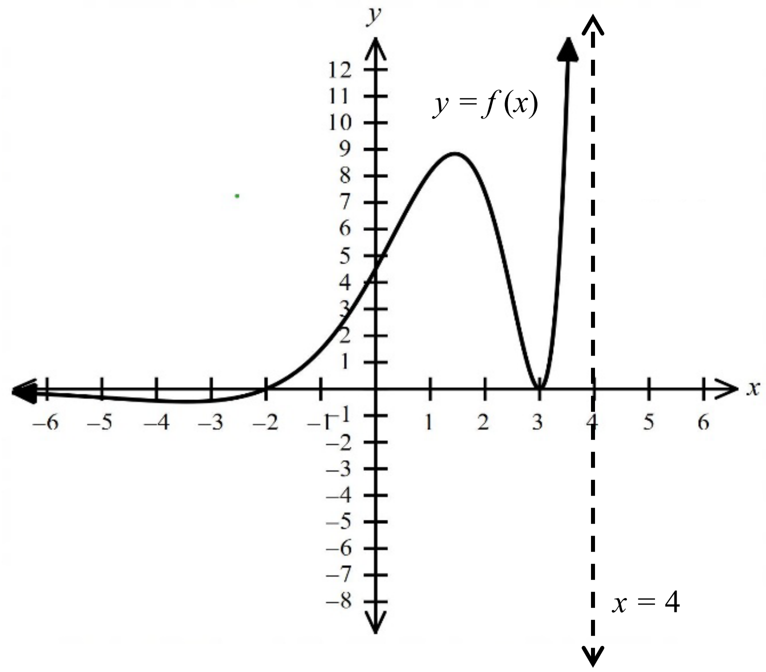
(i) Sketch the curve $y = f(x)$ clearly showing the coordinates of the end points. **2**

(ii) Find the equation of the inverse function $y = f^{-1}(x)$ and state its domain. **3**

(c) Solve $x^3 - 8x^2 + 5x + 50 = 0$, given that it has a root of multiplicity 2. **3**

Question 9 (12 marks) – Start a new booklet

(a) The graph of $y = f(x)$ is shown below, for the domain $(-6, 4)$.



(i) Draw a sketch of $y = |f(x)|$ showing all significant features. **1**

(ii) Draw a sketch of $y = \dfrac{1}{f(x)}$ showing all significant features. **3**

(b) Divide $x^4 + x^3 - 7x^2 + 6$ by $x^2 - 1$ using long division, giving your answer in the form $P(x) = A(x)Q(x) + R(x)$, where $A(x)$, $Q(x)$ and $R(x)$ are polynomials. **2**

(c) The position of a particle moving along the x-axis is given by $x = 10e^{-0.5t} + 15t - 20$, where t is the time in seconds and x is the displacement of the particle in metres.

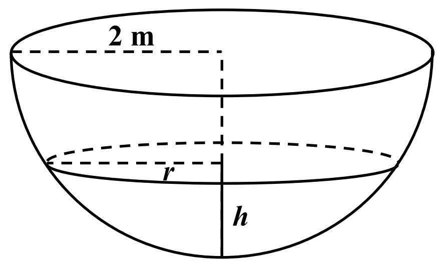
(i) Find the initial velocity and acceleration of the particle. **2**

(ii) When will the particle attain a velocity of 90% of its limiting velocity? **3**

(iii) Explain why the velocity is always increasing. **1**

Question 10 (14 marks) – Start a new booklet

(a) An open hemisphere tank has radius of 2 metres. The tank is being filled with water. When the water depth reaches 1 metre, the rate of change in the height of water is 50 cm/hr. Find the rate at which the top surface area of the water changing at this time. **3**



(b) The three roots of $x^3 + px + qx + r = 0$ are integers with a common difference of β.
That is, if the roots are x_1, x_2 and x_3 then $x_2 - x_1 = x_3 - x_2 = \beta$.
Show that $9pq = 27r + 2p^3$. **3**

(c) If $\tan\frac{\theta}{2} = \frac{a}{b}$, prove that $b\cos\theta + a\sin\theta = b$. **3**

(d) It is assumed that the population of a newly introduced species on an island will usually grow or decay in proportion to the difference between the current population P and the ideal population I, that is, $\frac{dP}{dt} = k(P - I)$ where k may be positive or negative integer.

(i) Prove that $P = I + Ae^{kt}$ is a solution to this equation, given time is in weeks. **1**

(ii) Initially 10000 animals are released. A census is taken 7 weeks later and again at 14 weeks, and the population grows to 12 000 and then 18 000.
Show that $k = \frac{1}{7}\ln(3)$, $A = 1000$ and $I = 9000$. **3**

(iii) Find the population after 21 weeks. **1**

End of Examination.

FORT STREET HIGH SCHOOL

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Solutions

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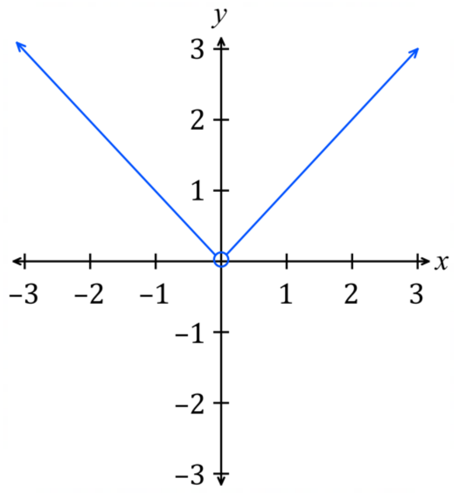

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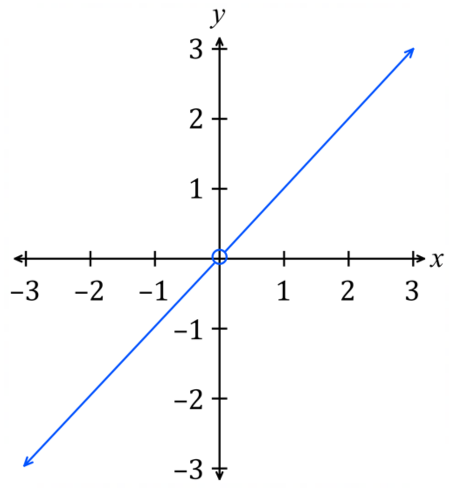

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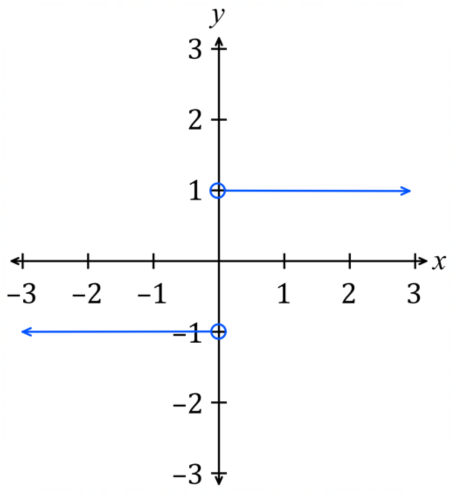

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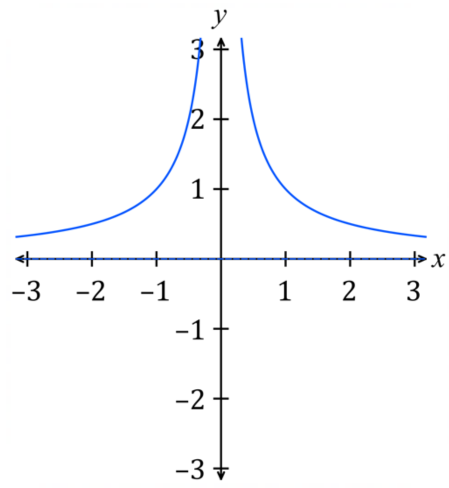

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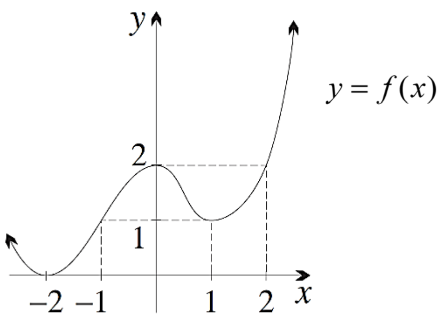

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Question 7 (13 marks) – Start a new booklet

(a) Solve $\frac{x-4}{x-1} > 3$. **3**

$$\frac{x-4}{x-1} \times (x-1)^2 > 3 \times (x-1)^2$$

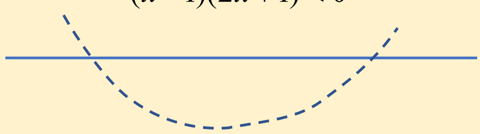
$$(x-1)(x-4) > 3(x-1)^2$$

$$(x-1)(x-4) - 3(x-1)^2 > 0$$

$$(x-1)\left[(x-4) - 3(x-1)\right] > 0$$

$$(x-1)(-2x-1) > 0$$

$$(x-1)(2x+1) < 0$$



$$\therefore -\frac{1}{2} < x < 1$$

1 – correct first step and some steps towards the final answer

1 – For justifying the solution

1 – For final answer

Marker's comments:

Most students knew to multiply by the square of the denominator and could also then bring all terms to one side and factorise. Students who expanded first found it hard to accurately factorise.

A large number of students did not draw the graph of the parabola and use this to determine the correct section of the number line for solution for x. Even-though some student wrote the correct answer they did not receive full mars if no graph was drawn, or no explanation provided.

(b) Let $P(x) = 3x^3 + 2x^2 - 7x + 2$.

(i) Show that $(x-1)$ is a factor of $P(x)$. **1**

$$P(1) = 3(1)^3 + 2(1)^2 - 7(1) + 2$$
$$= 3 + 2 - 7 + 2$$
$$= 0$$

1 – correct Solution

Marker's comments: Done very well. However, Students must be aware that simply substituting in x=1 and then saying P(1) is zero in many cases is an insufficient show. Each term should be evaluated to show that that P(1)=0.

(ii) Hence express $P(x)$ as a product of three linear factors. **2**

$$P(x) = 3x^3 + 2x^2 - 7x + 2$$
$$= (x-1)(3x^2 + 5x - 2) \text{ (By division transformation)}$$
$$= (x-1)(3x-1)(x+2)$$

1 – first factorisation

1 – correct full solution

Marker's comments:

Done quite well. If the division algorithm is used then most student could then correctly factorise the quadratic quotient to fully factorise.

Some student used sum and product of roots to determine the factors, which is fine, but many then forgot to make the leading term 3.

(c) The graph of $f(x) = (x-2)^2$; $[2,\infty)$ is shown below, along with the line $y = x$.

(i) Write the equation of the inverse function in the form $y = f^{-1}(x)$ and state its domain. **2**

Let $x = (y-2)^2$

$y - 2 = \pm\sqrt{x}$

$y = 2 + \sqrt{x}$ (ignoring the negative case due to limits on the range)

$D:[0,\infty)$

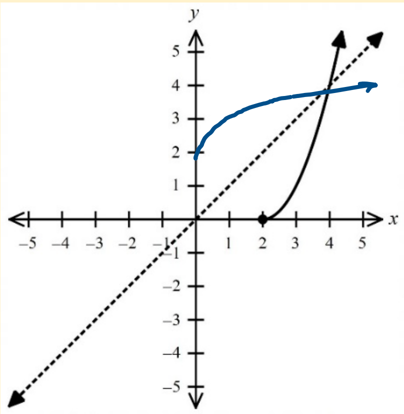
1 – equation of the inverse with reasoning for ignoring the negative case

1 – correct domain

Marker's comments:

Very few students received 2 marks. Most students did not explain sufficiently why the positive square root was taken. All that was required was to not that the range needed to be greater than 2. Most student correctly stated the domain. Many missed that part of the question.

(ii) Hence sketch the inverse function, including the original function and the line $y = x$ in your sketch. **2**



1 – Correct shape and y-intercept

1 – point of intersection on the line *y=x*

Marker's comments:

Quite well done. Most common error was to draw the inverse with a turning point instead of making it monotonically increasing.

(d) Prove that $\dfrac{2\cos\theta+1+\cos 2\theta}{2\cos\theta-1-\cos 2\theta}=\cot^2\dfrac{\theta}{2}$ **3**

1 – Simplifying LHS

1 – correct use of t-formula

1 – correct full proof

$$LHS=\frac{2\cos\theta+1+\cos 2\theta}{2\cos\theta-1-\cos 2\theta}$$

$$=\frac{2\cos\theta+1+\cos^2\theta-\sin^2\theta}{2\cos\theta-1-\cos^2\theta+\sin^2\theta}$$

$$=\frac{2\cos\theta+2\cos^2\theta}{2\cos\theta-2\cos^2\theta}$$

$$=\frac{1+\cos\theta}{1-\cos\theta}$$

Now let $t=\tan\dfrac{\theta}{2}$, then $\cos\theta=\dfrac{1-t^2}{1+t^2}$

Therefore:

$$LHS=\frac{1+\dfrac{1-t^2}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}}$$

$$=\frac{\dfrac{1+t^2+1-t^2}{1+t^2}}{\dfrac{1+t^2-1+t^2}{1+t^2}}$$

$$=\frac{2}{2t^2}$$

$$=\frac{1}{t^2}$$

$$=\frac{1}{\left(\tan\dfrac{\theta}{2}\right)^2}$$

$$=\cot^2\frac{\theta}{2}$$

$$=RHS$$

Marker's comments:

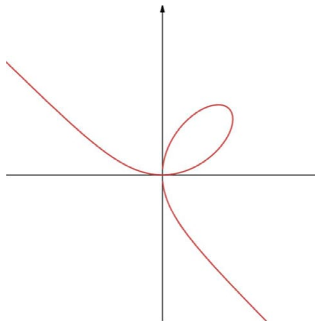
A difficult question, not many students received 3 marks. Many students who jumped straight into t-formulae could not get to the RHS. The most successful meth at was to make $\cos^2\theta$ substitution for both $\cos 2\theta$ terms. After simplifying, then use

t-formula.

Question 8 (11 marks) – Start a new booklet

(a) The graph to the right referred to as "the folium of Descartes", is for the curve defined parametrically by:

$$\begin{cases} x = \dfrac{3t}{1+t^3} \\ y = \dfrac{3t^2}{1+t^3} \end{cases}$$



(i) Find the simplest expression for $\dfrac{y}{x}$ in terms of t. **1**

$$\frac{y}{x} = \frac{\dfrac{3t^2}{1+t^3}}{\dfrac{3t}{1+t^3}}$$

$$= \frac{3t^2(1+t^3)}{3t(1+t^3)}$$

$$= t$$

1 – correct Solution

Marker's comments:

Generally answered well. A number of students did not simplify fully.

(ii) Deduce that the Cartesian equation of this curve is: $x^3 + y^3 = 3xy$. **2**

substitute $t = \dfrac{y}{x}$ into $x = \dfrac{3t}{1+t^3}$

$$x = \frac{3\dfrac{y}{x}}{1+\left(\dfrac{y}{x}\right)^3}$$

$$= \frac{\dfrac{3y}{x}}{\dfrac{x^3+y^3}{x^3}}$$

$$= \frac{3yx^2}{x^3+y^3}$$

$$x(x^3+y^3) = 3yx$$

$$x^3 + y^3 = 3xy \text{ (as required)}$$

1 – substitution leading to answer

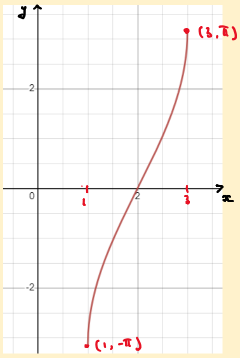
1 – completing the solution

Marker's comments:

Was not answered well. Many students proved that LHS=RHS, but this was not what the question asked. To receive any marks students needed to eliminate the parameter *t* from the equation.

(b) Consider the function $f(x) = 2\sin^{-1}(x-2)$

(i) Sketch the curve $y = f(x)$ clearly showing the coordinates of the end points. **2**



1 – Shape and position
1 – end points

Marker's comments:

Answered well. Students must label axis, show coord of end points, and be careful with dimensions and scales.

(ii) Find the equation of the inverse function $y = f^{-1}(x)$ and state its domain. **3**

$$x = 2\sin^{-1}(y-2)$$

$$\frac{x}{2} = \sin^{-1}(y-2)$$

$$y-2 = \sin\frac{x}{2}$$

$$f^{-1}(x) = 2 + \sin\frac{x}{2} \quad D:\{-\pi \le x \le \pi\}$$

1 – Steps leading to the solution
1 – complete correct solution
1 – Domain

Marker's comments:

Answered well. Some minor errors.

(c) Solve $x^3 - 8x^2 + 5x + 50 = 0$, given that it has a root of multiplicity 2. **3**

Let $P(x) = x^3 - 8x^2 + 5x + 50$

Then $P'(x) = 3x^2 - 8x + 5$

Let $P'(x) = 0$

Then $3x^2 - 8x + 5 = 0$

$(3x-1)(x-5) = 0$

Now $P(5) = 0$

Therefore $x = 5$ is the root of multiplicity 2.

By inspection $x^3 - 8x^2 + 5x + 50 = (x-5)^2(x+2)$

Therefore the solutions are $x = -2$ and $x = 5$.

1 – progress towards the final relation
1 – 1/3 not a solution
1 – final solution

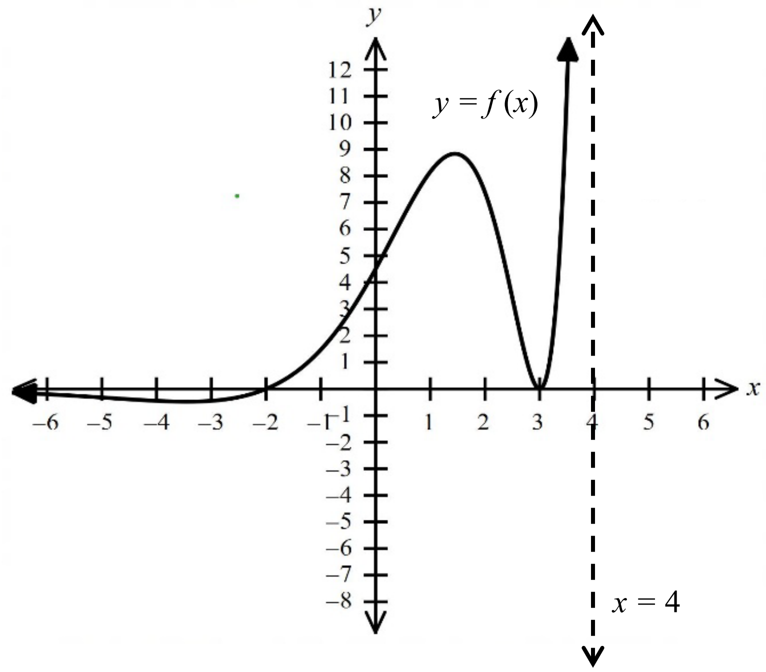
Marker's comments:

Generally, answer ed well by most students.

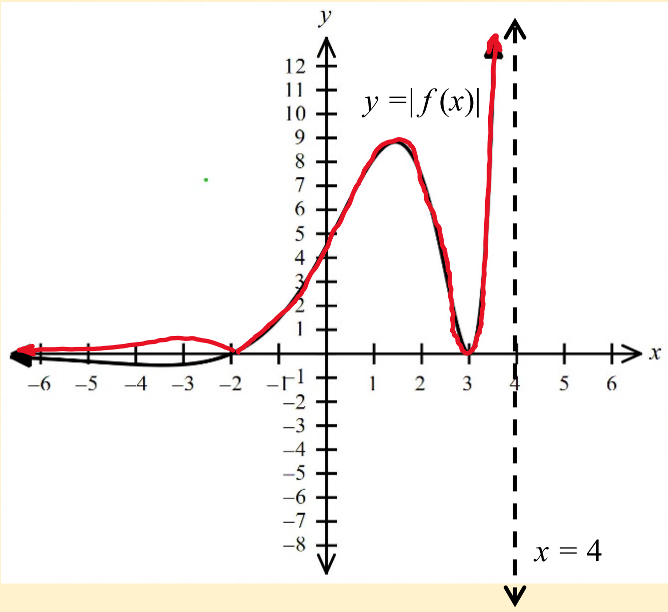
There was more than one strategy to answer this question.

Question 9 (12 marks) – Start a new booklet

(a) The graph of $y = f(x)$ is shown below, for the domain $(-6, 4)$.



(i) Draw a sketch of $y = |f(x)|$ showing all significant features. **1**

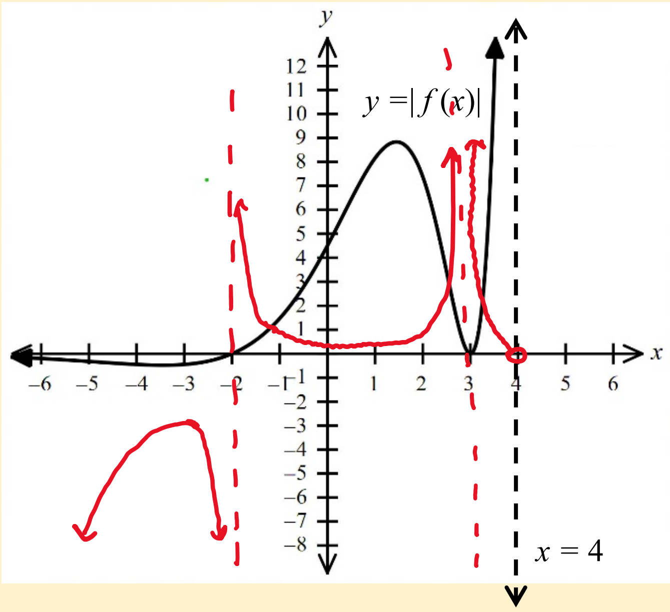


1 – correct solution

Marker's comments:

Mostly done well

(ii) Draw a sketch of $y = \dfrac{1}{f(x)}$ showing all significant features. **3**



1 – Correct general shape
2 – Correct solution with minor errors
3 – Complete solution with open circle at x=4, correct pts of intersections at x=1,-1 and y-int at around 1/4

Marker's comments:

Few students received 3/3 for this question.
The most common errors were:

- Missing the open circle when x=4
- Missing points of intersection when y=1, -1
- Sketching the graph wrong for the domain x<-2 (Some students did not consider the asymptote or they sketched it as if y->0 as x decreases)

(b) Divide $x^4 + x^3 - 7x^2 + 6$ by $x^2 - 1$ using long division, giving your answer in the form $P(x) = A(x)Q(x) + R(x)$, where $A(x)$, $Q(x)$ and $R(x)$ are polynomials. **2**

$x^2 + x - 6$

$x^2 - 1 \,\overline{)\, x^4 + x^3 - 7x^2 + 0x + 6}$

$x^4 \quad - x^2$

$x^3 - 6x^2 + 0x$

$x^3 \quad - x$

$-6x^2 + x + 6$

$-6x^2 \quad + 6$

x

$\therefore P(x) = (x^2 - 1)(x^2 + x - 6) + x$

1 – Incorrect final answer but correct division process

2 – Complete solution in the correct format

Marker's comments:

Mostly done well.

(c) The position of a particle moving along the x-axis is given by $x = 10e^{-0.5t} + 15t - 20$, where t is the time in seconds and x is the displacement of the particle in metres.

(i) Find the initial velocity and acceleration of the particle. **2**

$\dot{x} = -5e^{-0.5t} + 15$

$\ddot{x} = 2.5e^{-0.5t}$

Therefore, when $t = 0$,

Then $\dot{x} = -5e^{0} + 15$

$= -10\, m/s$

$\ddot{x} = 2.5e^{0}$

$= 2.5\ m/s^{2}$

1 – differentiation

1 –correct answer

Marker's comments:

Mostly done well

(ii) When will the particle attain a velocity of 90% of its limiting velocity? **3**

As $t \to \infty$, $e^{-0.5t} \to 0$

$\therefore$ As $t \to \infty$, $\dot{x} \to 15\,\text{ms}^{-1}$

Therefore, the limiting velocity is 15 ms^{-1}.

Let $\dot{x} = 0.9 \times 15$

Then $13.5 = -5e^{-0.5t} + 15$

$$e^{-0.5t} = \frac{3}{10}$$

$$-0.5t = \ln\left(\frac{3}{10}\right)$$

$$t = -2\ln\left(\frac{3}{10}\right)$$

$$t = \ln\left(\frac{100}{9}\right) \text{ seconds}$$

1 – calculation limiting velocity

1 – Steps towards final solution

1 – Exact time

Marker's comments:

Most common error was letting velocity v=0.9, rather than v=0.9*15=13.5. See the solutions to understand why the limiting velocity is 15. Students still received marks if the approximated their answer instead of giving an exact value.

(iii) Show that the velocity is always increasing. **1**

Since $\ddot{x} = 2.5e^{-0.5t}$, then acceleration is positive for all positive values of t.

And since velocity is 15 at $t = 0$, then velocity must be positive for all positive values of t.

1 – complete correct solution

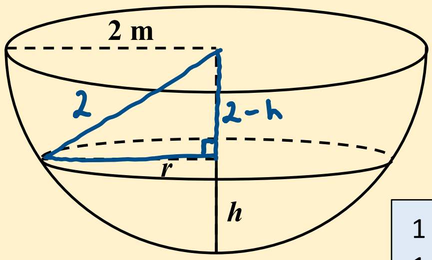
Marker's comments:

To receive marks students needed to SHOW the velocity is always increasing. Showing is an important skill required in the HSC course. To receive the mark students needed to SHOW the accel. is always positive, and the initial velocity is also positive and then make the conclusion that velocity is always positive. Both of these facts needed to be shown or adequately explained.

Question 10 (14 marks) – Start a new booklet

(a) An open hemisphere tank has radius of 2 metres. The tank is being filled with water. At what rate is the top surface area of the water changing when the depth is 1 metre, if at that point the rate of change in the height is 50 cm/hr. **3**

$$r^2 = (2)^2 - (2-h)^2$$
$$= 4 - 4 + 4h - h^2$$
$$= 4h - h^2$$
$$r = \left(4h - h^2\right)^{1/2}$$



$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dh} \times \frac{dh}{dt}$$
$$= (2\pi r) \times \frac{1}{2}\frac{(4-2h)}{(4h-h^2)^{1/2}} \times \frac{1}{2}$$
$$= \frac{\pi}{2}\frac{(4h-h^2)^{1/2}(4-2h)}{(4h-h^2)^{1/2}}$$
$$= \frac{\pi}{2}(4-2h)$$

1 – r in terms of h
1 – developing correct differential equation
1 – final answer

Marker's comments:

Poorly done.

Many students took similar triangles which was wrong

Therefore, when the depth is one metre, the rate of change of the surface area of the top of the water is π m/hr.

(b) The three roots of $x^3 + px^2 + qx + r = 0$ are integers with a common difference of β.
That is, if the roots are x_1, x_2 and x_3 then $x_2 - x_1 = x_3 - x_2 = \beta$.
Show that $9pq = 27r + 2p^3$. **3**

$$(\alpha - \beta) + \alpha + (\alpha + \beta) = -p$$

$$3\alpha = -p$$

$$\alpha = -\frac{p}{3} \quad --(1)$$

1 – roots equations

1 – correct substitution

1 – Rearrangement

$$\alpha(\alpha - \beta) + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = q$$

$$2\alpha^2 + \alpha^2 - \beta^2 = q$$

$$\alpha^2 - \beta^2 = q - 2\alpha^2 \quad --(2)$$

$$(\alpha - \beta)\alpha(\alpha + \beta) = -r$$

$$\alpha(\alpha^2 - \beta^2) = -r \quad --(3)$$

Substituting (1) and (2) into (3):

$$-\frac{p}{3}\left(q - \frac{2p^2}{9}\right) = -r$$

$$\frac{-pq}{3} + \frac{2p^3}{27} = -r$$

$$9pq = 27r + 2p^3$$

Marker's comments:

Poorly done.

Students couldn't identify that these three roots make an arithmetic sequence .

(c) If $\tan\frac{\theta}{2}=\frac{a}{b}$, prove that $b\cos\theta+a\sin\theta=b$. **3**

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan\theta=\frac{2\frac{a}{b}}{1-\frac{a^2}{b^2}} \quad\Rightarrow\quad \tan\theta=\frac{2ab}{b^2-a^2}$$

$$\therefore \sin\theta=\frac{2ab}{b^2+a^2} \quad\text{and}\quad \cos\theta=\frac{b^2-a^2}{b^2+a^2}$$

Therefore:

$$b\cos\theta+a\sin\theta=\frac{b^3-a^2b+2a^2b}{a^2+b^2}$$

$$=\frac{b(b^2+a^2)}{b^2+a^2}$$

$$=b$$

1 – finding sin and cos in terms of t

1 – Substitution

1 – correct rearrangement

Marker's comments:

Well done.

(d) It is assumed that the population of a newly introduced species on an island will usually grow or decay in proportion to the difference between the current population P and the ideal population I, that is, $\frac{dP}{dt}=k(P-I)$ where k may be positive or negative integer.

(i) Prove that $P=I+Ae^{kt}$ is a solution to this equation. **1**

$$P=I+Ae^{kt}$$

$$\frac{dP}{dt}=kAe^{kt}$$

$$=kI+kAe^{kt}-kI$$

$$=k(I+Ae^{kt}-I)$$

$$=k(P-I)$$

1 – complete correct solution

Marker's comments:

Well done.

(ii) Initially 10000 animals are released. A census is taken 7 weeks later and again at 14 weeks, and the population grows to 12 000 and then 18 000.

Show that $k = \frac{1}{7}\ln(3)$, $A = 1000$ and $I = 9000$. **3**

$P = I + Ae^{kt}$

$\left.\begin{array}{l} t = 0 \\ P = 10000 \end{array}\right\} \Rightarrow 10000 = I + A \quad --(1)$

$\left.\begin{array}{l} t = 7 \\ P = 12000 \end{array}\right\} \Rightarrow 12000 = I + Ae^{7k} \quad --(2)$

$\left.\begin{array}{l} t = 14 \\ P = 18000 \end{array}\right\} \Rightarrow 18000 = I + Ae^{14k} \quad --(3)$

Rearrange (1) to make I the subject and substitute the result in (2) and (3):

$\left.\begin{array}{l} 2000 = Ae^{7k} - A \\ 8000 = Ae^{14k} - A \end{array}\right] \div$

$\frac{1}{4} = \frac{e^{7k} - 1}{\left(e^{7k}\right)^2 - 1}$

$\frac{1}{4} = \frac{e^{7k} - 1}{\left(e^{7k} - 1\right)\left(e^{7k} + 1\right)}$

$\frac{1}{4} = \frac{1}{\left(e^{7k} + 1\right)}$

$e^{7k} + 1 = 4$

$e^{7k} = 3$

$k = \frac{1}{7}\ln(3)$

$A(e^{7k} - 1) = 2000$

$A = \frac{2000}{e^{\ln(3)} - 1}$

$= \frac{2000}{2}$

$A = 1000$

$I = 10000 - 1000$

$I = 9000$

1 – establishing three equations

1 – solving simultaneously correctly to find k

1 – Final solution for A and I

Marker's comments:

Poorly done.

Some students used the values of

A=1000 and I = 9000 to find k. They were required to prove this.

(iii) Find the population after 21 weeks. **1**

$P = 9000 + 1000e^{\frac{21}{7}\ln(3)}$

$= 9000 + 1000e^{\ln 27}$

$= 36000$

1 – complete correct solution

Marker's comments:

Well done.

End of Examination.