2024	PRELIMINARY YEARLY EXAMINATION

Mathematics Extension

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 4)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5 – 8)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

---

1 How many ways can 7 people be seated around a circular table if 2 particular people, Preksha and Chloe, do not wish to sit next to each other?

A. 480

B. 718

C. 2880

D. 3360

---

2 A parabola has the parametric equations $x = t - 2$ and $y = 2t^2 - 1$. What is the Cartesian equation of the parabola?

A. $y = 2x^2 + 4x - 1$

B. $y = 2x^2 + 4x + 1$

C. $y = 2x^2 + 8x - 7$

D. $y = 2x^2 + 8x + 7$

---

3 When $P(x)$ is divided by $x^2 + 5x - 6$, the remainder is the polynomial $R(x) = 2x - 5$. What is the remainder when $P(x)$ is divided by $(x - 1)$?

A. -7

B. -6

C. -3

D. 7

4 If $\frac{\pi}{2} < A < \pi$, and $\pi < B < \frac{3\pi}{2}$ with $\cos A = t$ and $\sin B = t$, then which of the following is equivalent to $\sin(A+B)$?

A. 0

B. 1

C. $2t^2 - 1$

D. $1 - 2t^2$

---

5 Which of the following is equivalent to $\frac{1-\cos 2x}{\sin 2x}$?

A. $1 - \cot 2x$

B. 1

C. $\cot x$

D. $\tan x$

---

6 Which of the following is an expression for $\frac{dy}{dt}$ if $y = \sqrt{25 - x^2}$ and $\frac{dx}{dt} = 2$?

A. $-4x\sqrt{25 - x^2}$

B. $-2x\sqrt{25 - x^2}$

C. $\frac{-4x}{\sqrt{25 - x^2}}$

D. $\frac{-2x}{\sqrt{25 - x^2}}$

7 The solutions to the equation $\sin 11\theta \sin 3\theta = \alpha$, will always be the same as the solutions to which of the following equations?

A. $\cos 14\theta - \cos 8\theta = \alpha$

B. $\cos 8\theta - \cos 14\theta = \alpha$

C. $\cos 14\theta - \cos 8\theta = 2\alpha$

D. $\cos 8\theta - \cos 14\theta = 2\alpha$

---

8 Let $(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$. If $P(x) = (4x^2 + ax - 1)^2$, what is the value of a?

A. 0

B. $\frac{1}{2}$

C. 1

D. 2

---

9 Lawrence is about to walk up a set of stairs with 8 steps. He can take either one or two steps at a time. In how many different ways can he walk up the set of stairs?

A. 5

B. 20

C. 34

D. 163

---

10 In how many ways can the letters of the word **COMPLEMENT** be arranged if the letter **M**'s are separated and the letter **E**'s are separated?

A. 544320

B. 545580

C. 564480

D. 584640

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 section of the writing booklet.

(a) Use a compound angle formula to find the exact value of $\cos 75°$. **2**

(b) (i) Find the value of k if $x-2$ is a factor of $P(x)=2x^3+x^2-kx+6$. **1**

(ii) Hence, solve $P(x)=0$. **2**

(c) (i) Show that $\cot\theta-\cot 2\theta=\operatorname{cosec} 2\theta$. **2**

(ii) Hence find the exact value of $\cot 15°$. **1**

(d) If α, β and γ are roots of $2x^3-5x^2+3x-5=0$, find the value of:

(i) $\alpha\beta+\alpha\gamma+\beta\gamma$ **1**

(ii) $\dfrac{1}{\alpha\gamma}+\dfrac{1}{\beta\gamma}+\dfrac{1}{\alpha\beta}$. **2**

(e) Solve $\dfrac{x+4}{x-3}\leq 2$. **3**

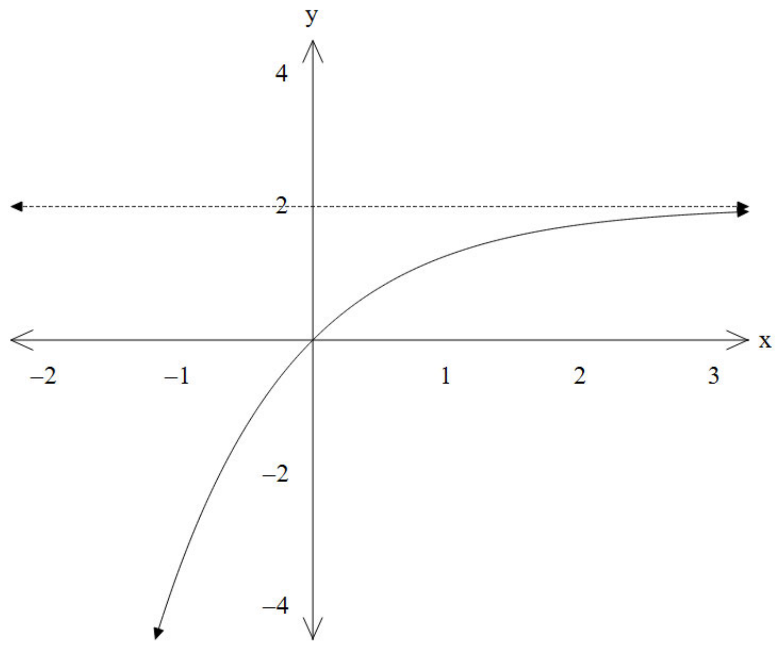
(f) The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is 1 + 2 + 3 = 6.

Nineteen two-digit numbers are randomly selected. Prove that at least two of them have the same digital sum. **2**

End of Question 11

Question 12 (16 marks) Use the Question 12 section of the writing booklet.

(a) The diagram shows the graph of $f(x) = 2 - 2e^{-x}$.



On the separate grids provided in your answer booklet, sketch the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $|y| = f(-x)$ **2**

(b) (i) Find the domain **AND** the range of the function $y = \sin^{-1}\left(\dfrac{x}{4} - 1\right)$. **2**

(ii) Sketch the graph of the function $y = \sin^{-1}\left(\dfrac{x}{4} - 1\right)$. **2**

(c) A polynomial is given by $P(x) = x^5 - 2x^4 - 6x^3 + 20x^2 - 19x + 6$.

(i) Prove that $x = 1$ is a root of $P(x)$ of multiplicity 3. **2**

(ii) Factorise $P(x)$ completely. **2**

(iii) Hence, solve $P(x) \geq 0$. **1**

(d) Prove that $\sin\left(2\sin^{-1} x + \cos^{-1} x\right) = \sqrt{1 - x^2}$. **3**

Question 13 (15 marks) Use the Question 13 section of the writing booklet.

(a) An egg at room temperature, $20°C$, is placed in a saucepan of boiling water which is maintained at $100°C$. When the egg has been in the boiling water for t minutes the internal temperature of the egg is $T°C$. The rate at which the internal temperature of the egg rises is proportional to the difference between the egg's internal temperature and that of the boiling water, so that T satisfies the equation $\frac{dT}{dt} = k(T - 100)$, where k is a constant.

(i) Show that $T = 100 + Ae^{kt}$ satisfies the equation. **1**

(ii) The internal temperature of the egg rises to $60°C$ after 10 minutes. Find the values of A **and** k. **2**

(iii) How long does it take the internal temperature of the egg to reach $90°C$? **1**

(iv) What would happen to the internal temperature of the egg if the egg was left in the water indefinitely? Justify your answer. **1**

(b) A function is defined by $f(x) = (x+2)^2 - 9$, for $-2 \le x \le 2$.

(i) Find the equation of the inverse function $y = f^{-1}(x)$. **1**

(ii) On the same number plane, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the coordinates of the intercepts on the coordinate axes and the endpoints. **3**

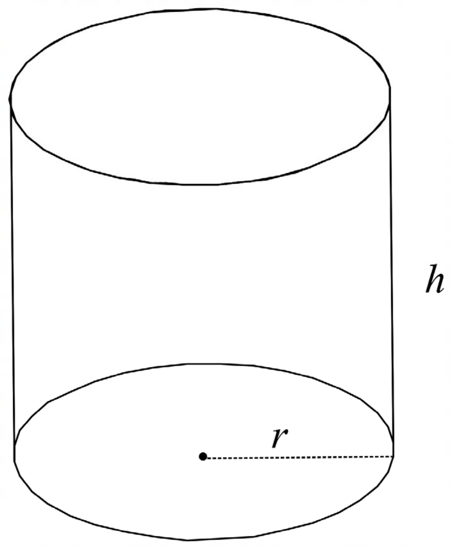
(iii) Find the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$, giving the answer in simplest exact form. **2**

(c) (i) Write down the expansion of $x(1+x)^n$ in ascending powers of x. **1**

(ii) Hence show that $2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + n\,{}^nC_{n-1} = (n+2)(2^{n-1} - 1)$. **3**

Question 14 (13 marks) Use the Question 14 section of the writing booklet.

(a) A balloon in the shape of a cylinder with height h and radius r, expands so that h is always proportional to r, that is $h = kr$ for some constant k. When $r = 4$ cm. the volume is increasing at the rate of $0.2\ \text{cm}^3\text{s}^{-1}$.
[The formula for the volume of a cylinder is $V = \pi r^2 h$ and the formula for the surface area of a cylinder is $A = 2\pi r(r+h)$]



(i) Show that when $r = 4$ cm, the rate of change of the radius is given by $\frac{dr}{dt} = \frac{1}{240\pi k}$ cms^{-1}. **2**

(ii) If the surface area of the balloon is increasing at the rate of $0.1\ \text{cm}^2\text{s}^{-1}$ when $r = 4$ cm, find the value of the constant of proportionality, k. **3**

(b) Consider the word **PHOTOGRAPHS**.

(i) How many arrangements of the letters of the word are possible? **2**

(ii) What is the probability that there are exactly four letters between the two **H**'s when the letters of the word **PHOTOGRAPHS** are arranged in a straight line? **2**

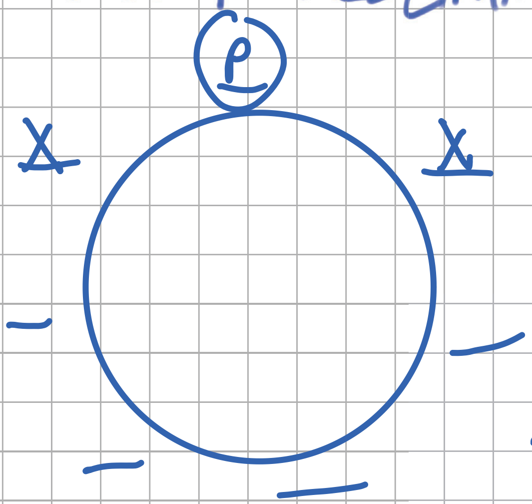
(c) (i) Show that ${}^{n+1}C_r - {}^{n}C_r = {}^{n}C_{r-1}$, for $r = 1, 2, 3, \ldots, n$. **2**

(ii) Hence find the value of ${}^{3}C_2 + {}^{4}C_2 + {}^{5}C_2 + \ldots + {}^{100}C_2$. **2**

End of Paper

X1 2024 PRELIMINARY EXAM SOLUTIONS

1.



sit Preksha in 1 way

sit Chloe in 4 ways

remaining people in 5! ways

$1 \times 4 \times 5!$

$= 480$

$\therefore$ A

2. $x = t - 2$

$t = x + 2$

$y = 2t^2 - 1$

$= 2(x+2)^2 - 1$

$y = 2x^2 + 8x + 7$

$\therefore$ D

3. $P(x) = (x^2 + 5x - 6)Q(x) + (2x - 5)$

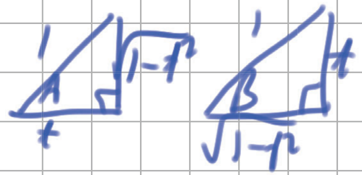
$P(1) = 0\,Q(1) + 2 - 5$

$P(1) = -3$

$\therefore$ C

4. $\sin(A+B)$

$\cos A = t \quad \sin B = t$



$\sin A = \sqrt{1-t^2} \quad \sin B = t$

$\cos A = t \quad \cos B = -\sqrt{1-t^2}$

$= \sin A \cos B + \cos A \sin B$

$= \sqrt{1-t^2}\left(-\sqrt{1-t^2}\right) + t \times t$

$= -\left(1-t^2\right) + t^2$

$= -1 + 2t^2$

$\therefore$ C

5. $\dfrac{1-\cos 2x}{\sin 2x}$

$\cos 2x = 1 - 2\sin^2 x$

$\sin 2x = 2\sin x \cos x$

$= \dfrac{2\sin^2 x}{2\sin x \cos x}$

$= \tan x$

$\therefore$ D

6. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$

$= \dfrac{1}{2}\left(25-x^2\right)^{-\frac{1}{2}}(-2x) \times 2$

$= \dfrac{-2x}{\sqrt{25-x^2}}$ $\therefore$ D

7. $\sin 11\theta \sin 3\theta = \alpha$

$\dfrac{1}{2}\left(\cos(11\theta - 3\theta) - \cos(11\theta + 3\theta)\right) = \alpha$

$\cos 8\theta - \cos 14\theta = 2\alpha$

$\therefore$ D

8. $(x-1)(4x^2+ax-1)^2 = 16x^5-20x^3+5x-1$

when $x=\frac{1}{2}$ $\left(-\frac{1}{2}\right)\left(1+\frac{a}{2}-1\right)^2 = 16\times\frac{1}{32}-\frac{20}{8}+\frac{5}{2}-1$

$-\frac{a^2}{8} = \frac{1}{2}-\frac{5}{2}+\frac{5}{2}-1$

$-\frac{a^2}{8} = -\frac{1}{2}$

$a^2 = 4$

$a = \pm 2$

when $x=2$, $(16+2a-1)^2 = 361$

$2a+15 = \pm 19$

$2a = 4$ or $2a = -34$

$a = 2$ or $a = -17$

$\therefore a = 2$. $\therefore$ D

9.

Double steps	Single steps		
0	8	SSSSSSSSSS	= 1 way
1	6	DSSSSSS	= 7 ways
2	4	DDSSSS	$= \frac{6!}{4!2!} = 15$ ways
3	2	DDDSS	$= \frac{5!}{3!2!} = 10$ ways
4	0	DDDDD	= 1 way

Total $= 1+7+15+10+1$

$= 34$ ways

$\therefore$ C

10. C O M P L E N T
M E

Total − M's and E's together − Ms together (E not together)
− E's together (Ms not together)

Ms and E's together

$\boxed{MM}\ \boxed{EE}$ C O P L N T

ways $= 8!$

M's together Es separate

_ $\boxed{MM}$ _ C _ O _ P _ L _ N _ T _

ways $= 7! \times {}^{8}C_2 = 141120$

similarly for E's together, M's separate

ways $= 141120$

$\therefore$ Possibilities $= \dfrac{10!}{2!2!} - 8! - 2 \times 141120$

$= 584640$

$\therefore$ D

Alternatively,

# of ways = total − E's together − M's together + E's <u>and</u> M's together

$= \dfrac{10!}{2!2!} - \left(\dfrac{9!}{2}\right) \times 2 + 8!$

C O $\boxed{MM}$ P L $\boxed{EE}$ N T

$= 584640$

11. a) $\cos 75^\circ = \cos(45+30)^\circ$

$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

② correct solution

① correct expansion of $\cos(45+30)^\circ$ or equivalent

b) i) $P(2)=0$

$0 = 16+4-2k+6$

$0 = 26-2k$

$2k = 26$

$k = 13$

① correct answer

b ii) $2x^3+x^2-13x+6=0$

$(x-2)(2x^2+5x-3)=0$

$(x-2)(2x-1)(x+3)=0$

$\therefore x = -3, \frac{1}{2}, 2$

OR Let roots be $\alpha, \beta, 2$.

$\Sigma\alpha$: $\alpha+\beta+2 = -\frac{1}{2}$

$\Sigma\alpha\beta\gamma$: $2\alpha\beta = -3$

$\alpha\beta = -\frac{3}{2}$

$\alpha + \frac{-3}{2\alpha} = -\frac{5}{2}$

② correct solution

① factorises $P(x)$ into two terms

or ① uses sum of product of roots and attempts to solve simultaneously

$$2\alpha^2 - 3 = -5\alpha$$
$$2\alpha^2 + 5\alpha - 3 = 0$$
$$(2\alpha - 1)(\alpha + 3) = 0$$
$$\alpha = \tfrac{1}{2} \text{ or } \alpha = -3$$
$$\therefore \alpha = -3, \tfrac{1}{2} \text{ and } 2$$

11 c) i) $\cot\theta - \cot 2\theta$

$$= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta}$$

$$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta} - \frac{\cos 2\theta}{\sin 2\theta}$$

$$= \frac{2\cos^2\theta - \cos 2\theta}{\sin 2\theta} \quad (\sin 2\theta = 2\sin\theta\cos\theta)$$

$$= \frac{1}{\sin 2\theta} \quad (\text{since } \cos 2\theta = 2\cos^2\theta - 1)$$

$$= \csc 2\theta$$

2 correct proof

1 uses a double angle result correctly

OR 1 expresses $\cot\theta$ and $\cot 2\theta$ in terms of t results

OR let $t = \tan\frac{2\theta}{2}$
$= \tan\theta$

$$\cot\theta - \cot 2\theta = \frac{1}{t} - \frac{1-t^2}{2t}$$
$$= \frac{2-(1-t^2)}{2t}$$
$$= \frac{1+t^2}{2t}$$
$$= \csc 2\theta$$

ii) $\cot\theta - \cot 2\theta = \csc 2\theta$

1 correct answer (must use (i))

let $\theta = 15^\circ$

$$\cot 15^\circ = \csc 30^\circ + \cot 30^\circ$$
$$= 2 + \sqrt{3}$$

11 d) i) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{2}$

① correct answer

ii) $\frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\beta}$

$= \frac{\beta}{\alpha\beta\gamma} + \frac{\alpha}{\alpha\beta\gamma} + \frac{\gamma}{\alpha\beta\gamma}$

$= \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$= \frac{\frac{5}{2}}{\frac{5}{2}}$

$= 1$

② correct solution

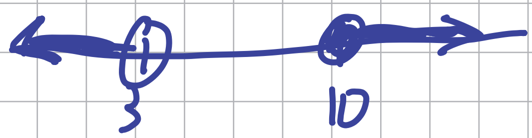
① finds $\alpha+\beta+\gamma$ OR $\alpha\beta\gamma$

11e) $\frac{x+4}{x-3} \leq 2 \qquad x \neq 3$

$x+4 = 2(x-3)$

$x+4 = 2x-6$

$x = 10$



$x < 3$ or $x \geq 10$

③ correct solution

② finds both critical values

① finds $x = 10$ as a critical value

or ① multiplies both sides by $(x-3)^2$

11 f)

	Digital sums
10	1
11	2
⋮	⋮
99	18

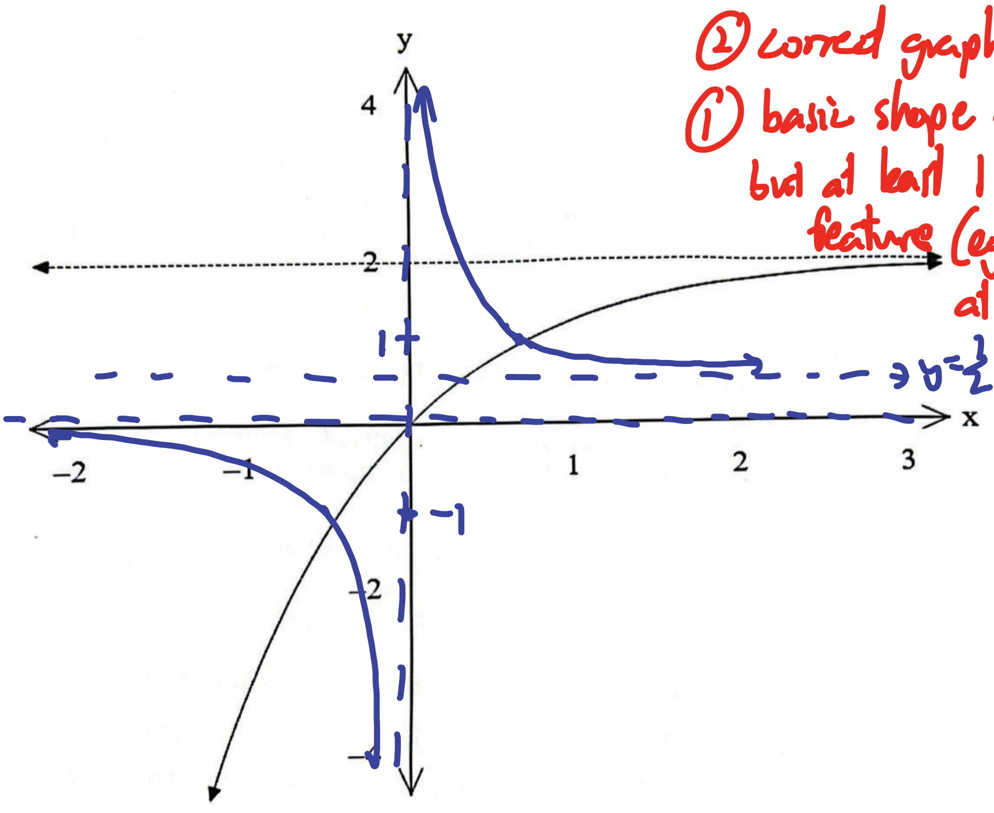
② correct solution

① identifies possible digital sums

Possible digital sums = 18

since 19 two digit numbers are chosen, by the pigeon hole principle there must be at least two of them that have the same digital sum

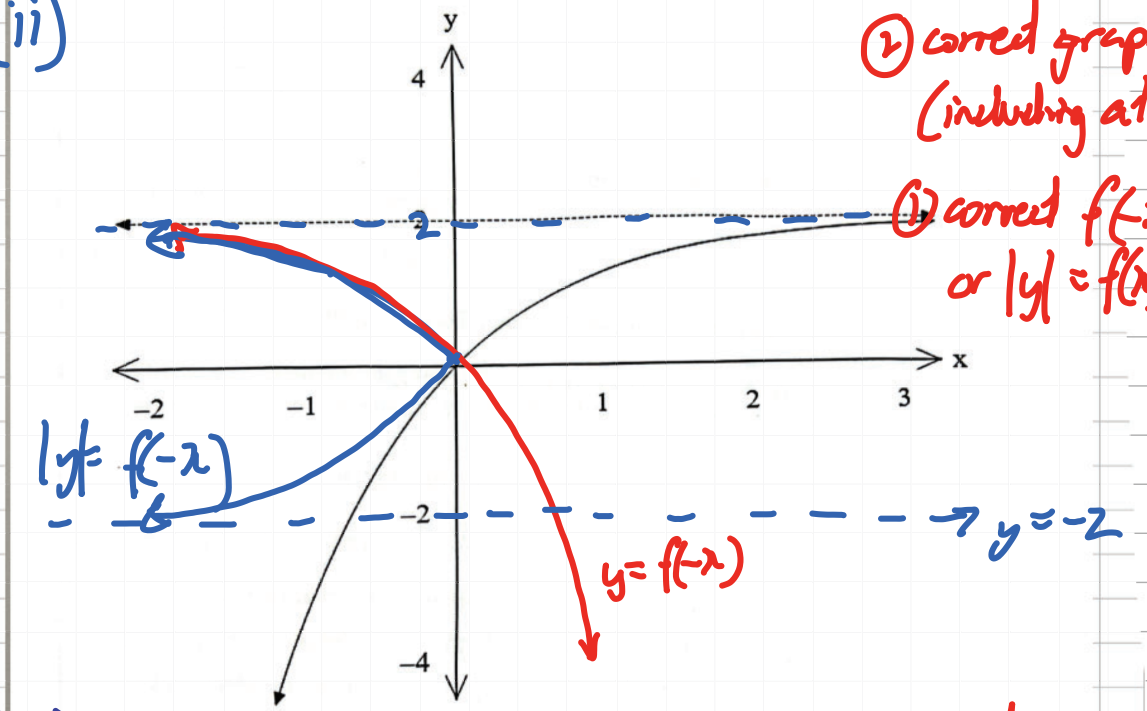
12 a i)



② correct graph

① basic shape correct but at least 1 missing feature (eg asymptote at $y=\frac{1}{2}$)

12 a(ii)



② correct graph (including at vertex)

① correct f(−x) or |y| = f(x)

12 b) i) D: $-1 \le \frac{x}{4} - 1 \le 1$

$0 \le \frac{x}{4} \le 2$

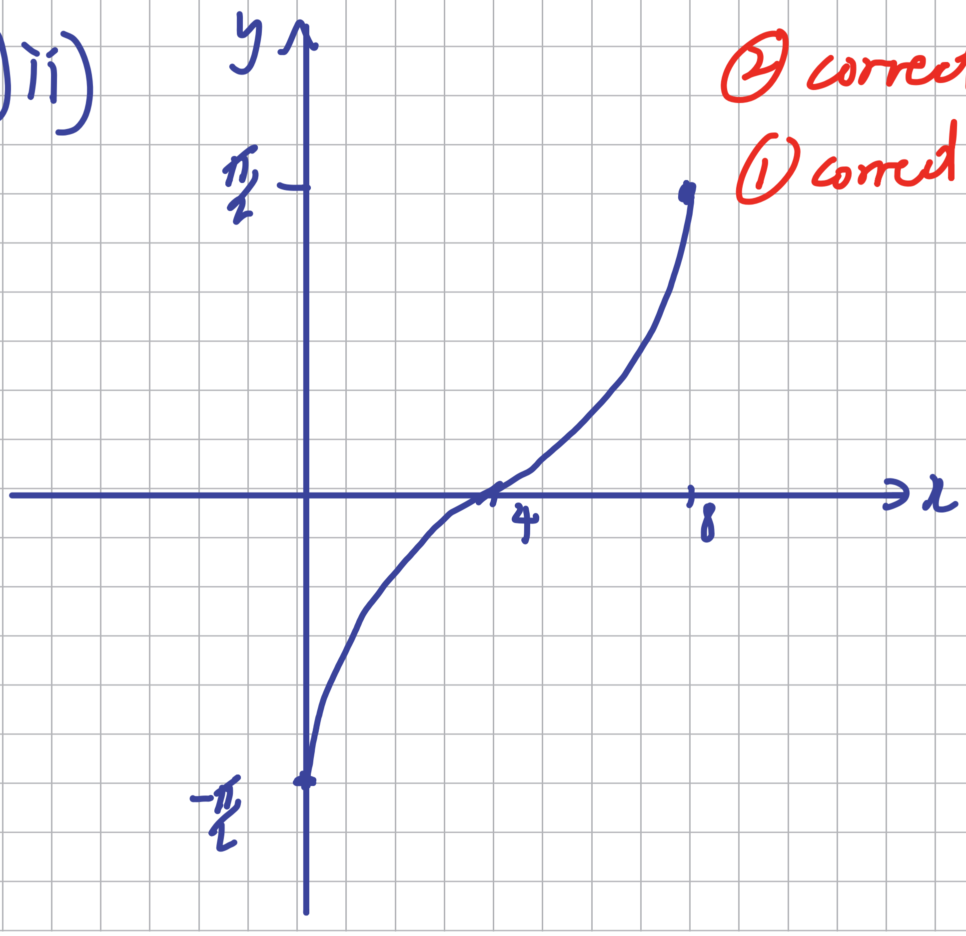
$0 \le x \le 8$

R: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

② correct domain and range

① correct domain or range

12 b) ii)



② correct graph

① correct basic shape

12 c i) $P(x) = x^5 - 2x^4 - 6x^3 + 20x^2 - 19x + 16$

$P'(x) = 5x^4 - 8x^3 - 18x^2 + 40x - 19$

$P''(x) = 20x^3 - 24x^2 - 36x + 40$

$P'''(x) = 60x^2 - 48x - 36$

$P(1) = 1 - 2 - 6 + 20 - 19 + 16$
$= 0$

$P'(1) = 5 - 8 - 18 + 40 - 19$
$= 0$

$P''(1) = 20 - 24 - 36 + 40$
$= 0$

$P'''(1) = -24$ $\therefore$ not multiplicity 4.

$\therefore$ Root of multiplicity 3.

② correct solution

① shows $P(1), P'(1), P''(1)$ and $P'''(1)$ are 0.

12 c ii) $P(x) = (x-1)^3(x-\alpha)(x-\beta)$

Sum of roots $1+1+1+\alpha+\beta = 2$

$\alpha+\beta = -1$

Product of roots $1 \times 1 \times 1 \times \alpha \times \beta = -6$

$\alpha\beta = -6$

$\alpha = -3,\ \beta = 2$

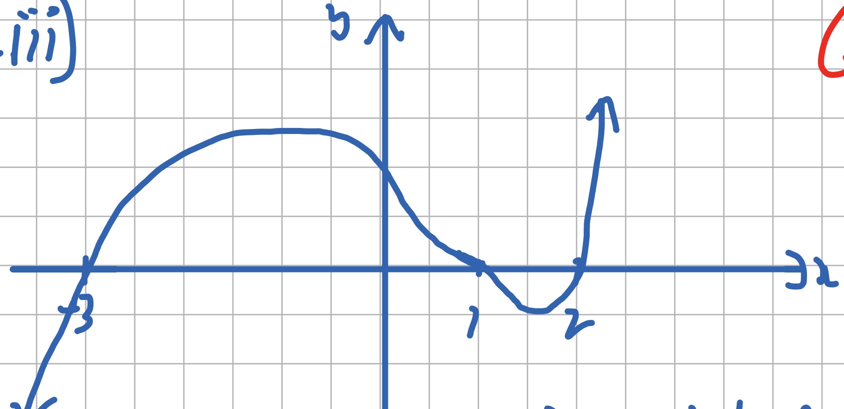
$\therefore P(x) = (x-1)^3(x+3)(x-2)$

② correct answer

① obtains x^2+x-6 as a factor by long division or inspection

or ① obtains $\alpha+\beta=-1$ and $\alpha\beta=-6$

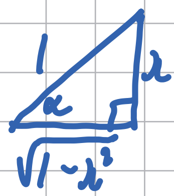
12 c iii)



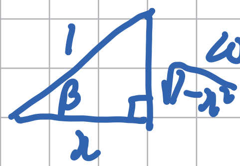
$\therefore -3 \le x \le 1$ or $x \ge 2$

① correct answer

12d) let $\alpha = \sin^{-1} x$, $\beta = \cos^{-1} x$



$\sin\alpha = x$



$\cos\beta = x$

$\sin(2\sin^{-1}x + \cos^{-1}x)$

$= \sin(2\alpha + \beta)$

$= \sin 2\alpha \cos\beta + \cos 2\alpha \sin\beta$

$= 2\sin\alpha\cos\alpha\cos\beta + (1 - 2\sin^2\alpha)\sin\beta$

$= 2x^2\sqrt{1-x^2} + (1-2x^2)\sqrt{1-x^2}$

$= \sqrt{1-x^2}\,(2x^2 + 1 - 2x^2)$

$= \sqrt{1-x^2}$ as req'd.

③ correct solution

② uses correct expression for expansion of $\sin(2\alpha+\beta)$ or equivalent and expresses one of the terms in x

(ie $\cos\alpha = \sqrt{1-x^2}$ or $\sin\beta = \sqrt{1-x^2}$

① writes correct expression for $\sin(2\alpha+\beta)$

OR finds $\cos\alpha = \sqrt{1-x^2}$ or $\sin\beta = \sqrt{1-x^2}$

① correct proof

13a) i) $T = 100 + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$

$= k(100 + Ae^{kt} - 100)$

$= k(T - 100)$ (using $T = 100 + Ae^{kt}$)

ii) When $t = 0$, $T = 20$

$20 = 100 + Ae^0$

$A = -80$

$\therefore T = 100 - 80e^{kt}$

when $t = 10$, $T = 60$

$60 = 100 - 80e^{10k}$

$80e^{10k} = 40$

$e^{10k} = \frac{1}{2}$

$10k = \ln\left(\frac{1}{2}\right)$

$k = \frac{-\ln 2}{10}$

$\therefore A = -80$ and $k = \frac{-\ln 2}{10}$

② correct solution

① finds value of A.

iii) When $T = 90$ ① correct answer

$90 = 100 - 80e^{kt}$

$80e^{kt} = 10$

$e^{kt} = \frac{1}{8}$

$kt = \ln\left(\frac{1}{8}\right)$

$t = \frac{-10}{\ln 2} \times -\ln 8$

$t = 30$

$\therefore$ Takes 30 minutes for internal temperature to reach 90°C.

iv) $T = 100 - 80e^{kt}$ ① correct answer (with justification)

As $t \to \infty$, $e^{-\frac{1}{10}\ln 2 t} \to 0$

$\therefore T \to 100 - 80 \times 0 = 100$

$\therefore$ Internal temperature approaches 100°C

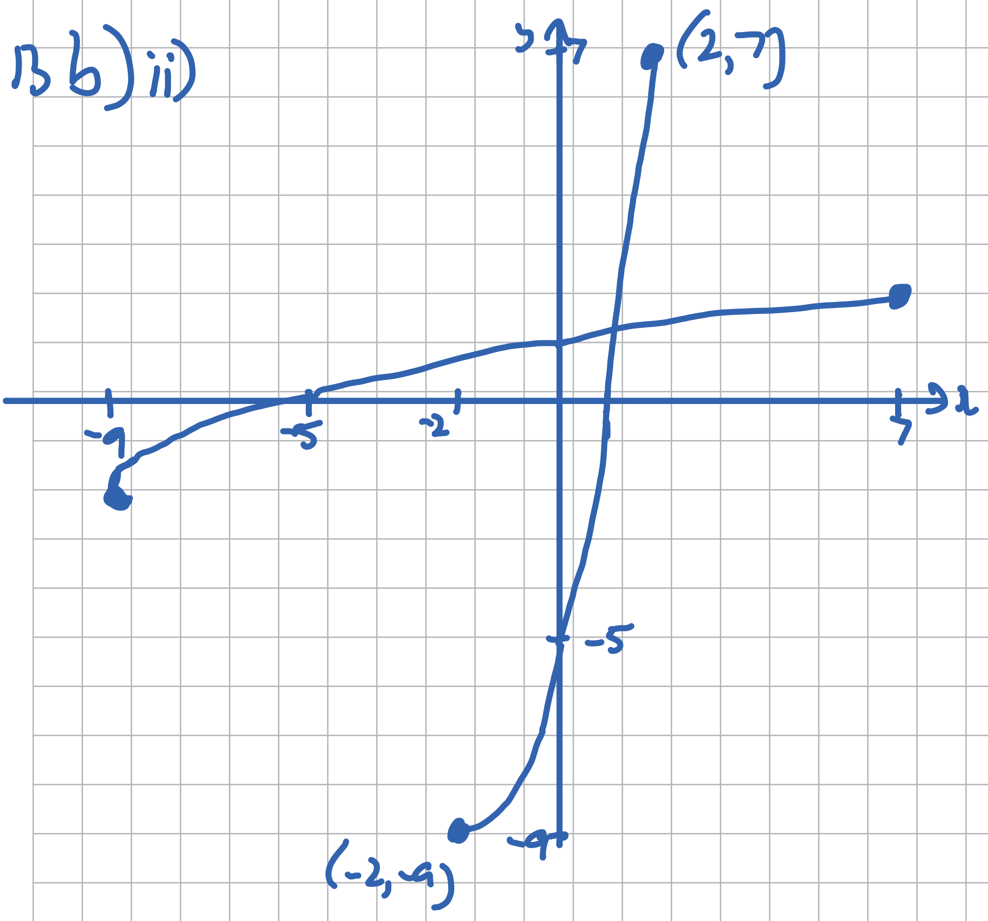
13 b) i) $f^{-1}(x)$: $x = (y+2)^2 - 9$ $-2 \leq y \leq 2$ ① correct equation

$(y+2)^2 = x + 9$

$y + 2 = \pm\sqrt{x+9}$

$f^{-1}(x) = -2 + \sqrt{x+9}$

13 b) ii)



③ Correct graphs with intercepts and end-points (including shape)

② one correct graph with endpoints and intercepts

② Both graphs correct but missing or incorrect endpoints and/or intercepts

① one correct graph with missing and/or incorrect endpoints or intercepts

13 b) iii)

$x = (x+2)^2 - 9$

$x^2 + 4x - 5 = x$

$x^2 + 3x - 5 = 0$

$x = \dfrac{-3 \pm \sqrt{9+20}}{2}$

$x = \dfrac{-3 \pm \sqrt{29}}{2}$

But $-2 \le x \le 2$, $\therefore x = \dfrac{\sqrt{29} - 3}{2}$

or

$x = -2 + \sqrt{x+9}$

$x + 2 = \sqrt{x+9}$

$x^2 + 4x + 4 = x + 9$

$x^2 + 3x - 5 = 0$

$x = \dfrac{-3 \pm \sqrt{29}}{5}$

② correct solution

① obtains two x values but doesn't eliminate the incorrect solution

① correct expansion

13 c i) $x(1+x)^n = x\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n\right]$

$= x+\binom{n}{1}x^2+\binom{n}{2}x^3+\dots+\binom{n}{n-1}x^n+\binom{n}{n}x^{n+1}$

ii) Differentiate wrt x

③ correct proof

② differentiates both sides & substitutes $x=1$

① differentiates one side and substitutes $x=1$

$(1+x)^n \cdot 1 + x\, n(1+x)^{n-1} = 1+2\,{}^nC_1 x+3\,{}^nC_2x^2+\dots+n\binom{n}{n-1}x^{n-1}+(n+1)x^n$

Let $x=1$

$2^n + n\times 2^{n-1} = 1+2\,{}^nC_1+3\,{}^nC_2+\dots+n\,{}^nC_{n-1}+(n+1)$

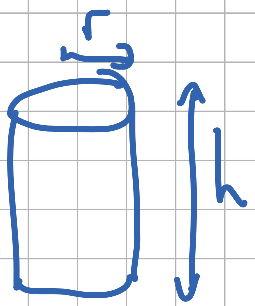
$\therefore 2\,{}^nC_1+3\,{}^nC_2+\dots+n\,{}^nC_{n-1} = 2^n+n2^{n-1}-1-n-1$

$= 2^n+n2^{n-1}-(n+2)$

$= 2^{n-1}(2+n)-1(n+2)$

$= (n+2)(2^{n-1}-1)$

14 a) i)



$h = kr$

② correct proof

① obtains $\frac{dV}{dr}$

$V = \pi r^2 h$

But $h = kr$

$\therefore V = \pi k r^3$

$\frac{dV}{dr} = 3\pi k r^2$

$\frac{dV}{dt} = 4$ when $r = 4$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 3\pi k r^2 \times \frac{dr}{dt}$$

when $r = 4$

$$\frac{1}{5} = 48\pi k \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{240\pi k}$$

ii) $A = 2\pi r^2 + 2\pi r h$ ③ correct solution

$= 2\pi r(r+h)$ ② obtains correct equation in terms of k

$= 2\pi r(r+kr)$

$A = 2\pi r^2(k+1)$ ① finds $\frac{dA}{dr}$

$$\frac{dA}{dr} = 4\pi(k+1)r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

when $r = 4$, $\frac{dA}{dt} = 0.1$

$$\frac{1}{10} = 16\pi(k+1) \times \frac{1}{240k\pi}$$

$$\frac{3}{2} = \frac{k+1}{k}$$

$$3k = 2k + 2$$

$$k = 2$$

14b) i) P H O T G R A S
P H O

Arrangements $= \frac{11!}{2!2!2!}$

$= 4989600$

(2) correct solution
(1) calculates 11!

b ii)

H _ _ _ _ H _ _ _ _ _
* * * * * *

Second H can go in 6 positions

Remaining letters can be arranged in $\frac{9!}{2!2!}$ ways

Total arrangements $= 6 \times \frac{9!}{2!2!}$

$= 544320$

$\therefore$ Probability $= \frac{544320}{4989600}$

$= \frac{6}{55}$

(2) correct solution
(1) considers possible positions of H's
OR (1) determines ways of arranging other 9 letters
(must have explained their calculations)

c) $^{n+1}C_r - {}^nC_r = \frac{(n+1)!}{(n+1-r)!r!} - \frac{n!}{(n-r)!r!}$

$= \frac{(n+1)!}{(n+1-r)!r!} - \frac{(n-r+1)n!}{(n-r+1)!r!}$

$= \frac{(n+1)n! - (n-r+1)n!}{(n-r+1)!r!}$

$= \frac{n!(n+1-(n-r+1))}{(n-r+1)!r!}$

$= \frac{n!r}{(n-r+1)!r(r-1)!}$

(2) correct solution
(1) obtains a common denominator

$$= \frac{n!}{(n-r+1)!(r-1)!}$$

$$= \frac{n!}{(n-(r-1))!(r-1)!}$$

$$= {}^{n}C_{r-1}$$

ii) ${}^{3}C_2 + {}^{4}C_2 + {}^{5}C_2 + \dots {}^{100}C_2$

② correct solution

① uses identity in (i)

$= ({}^{4}C_3 - {}^{3}C_3) + ({}^{5}C_3 - {}^{4}C_3) + ({}^{6}C_3 - {}^{5}C_3) + \dots + ({}^{101}C_3 - {}^{100}C_3)$

$= {}^{101}C_3 - {}^{3}C_3$

$= 166649$