Mathematics Extension 2

HSC Marking Feedback 2018

Question 11

Skills addressed:

- simple operations involving complex numbers (ai)
- division of complex numbers involving a conjugate of a complex number (aii)
- solving problems involving multiple roots of a polynomial (b)
- integrate expressions that involve the use of partial fractions (c)
- performing vector rotation (di)
- addition of two vectors (dii)
- applying the properties of quadrilaterals (diii)
- applying the theorems involved in circle geometry (e).

In better responses, students:

- correctly collect the real and imaginary parts (ai)
- find a conjugate of a complex number and 'realise' the denominator in division by a complex number (aii)
- show the ability to differentiate and use the double root to find the value of $\boldsymbol{a}$. They could then find $\boldsymbol{b}$ and $\boldsymbol{r}$ by substituting $\boldsymbol{a}$ into relevant equations (b)
- use the sum and product of roots to find the values of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{r}$. This led to solving simple linear equations (b)
- equate expressions correctly and use simultaneous equations, or substitution, to find the unknowns and then find the correct primitive (c)
- multiply $\boldsymbol{u}$ by $\boldsymbol{i}$ to find $\boldsymbol{w}$ (di)
- add $\boldsymbol{w}$ and $\boldsymbol{u}$ to find $\boldsymbol{v}$ (dii)
- recognise that the vector representing the complex number $\frac{\boldsymbol{w}}{\boldsymbol{v}}$ was the diagonal of the square and that $\mathbf{arg}(\frac{\boldsymbol{w}}{\boldsymbol{v}})$ was half of the vertex angle (diii)
- use clear and concise reasoning to describe relationships between angles in the circles (e)
- find an expression for $\boldsymbol{d}$ in the right-angle triangle $\boldsymbol{ABC}$ involving angle $\boldsymbol{B}$, then convert to an expression in terms of angle $\boldsymbol{D}$, rather than the other way around (e)
- find that the angle at the centre is twice the size of angle $\boldsymbol{D}$ by giving the correct reason then using trigonometric ratios in triangle $\boldsymbol{AOC}$ (e).

Areas for students to improve include:

- practising the multiplication and addition of complex numbers as the common error in adding the imaginary part was $-2i + 3i = -i$ (ai)
- practising the four basic operations involving complex numbers (aii)
- taking care with algebraic manipulation to eliminate errors in solving simultaneous equations, including the misuse of positive and negative signs (b)
- practising factorisation by inspection rather than using long division to find r by realising that 4 is the double root (b)

- practising the integration of rational functions involving the use of partial fractions. The common errors involved finding the incorrect value for c which led to a more difficult integration or having to integrate $-\frac{1}{x^2-3}$ instead of $-\frac{x}{x^2-3}$. (c)
- practising the rotation of a vector when working with complex numbers (di)
- using the addition of two vectors when working with complex numbers (dii)
- using the addition of two vectors when working with complex numbers (diii)
- practising circle geometry questions that involve the use of trigonometric rules. Common errors included: incorrect labelling of angles; assuming that triangle ABC was a right-angled without reason; not explaining the appropriate property being used at each step (e).

Question 12

Skills addressed:

- simplify algebraic expressions before integrating (a)
- using implicit differentiation to compute the derivative of curves given in implicit form (bi)
- integrating rational functions by completing the square in the quadratic denominator (c)
- using a given graph and the properties of the absolute value function to produce the desired graph (di)
- recognising that the graph of $y = (f(x))^2 \geq 0$ for all x in the domain (dii)
- writing the equation of the curve $y = x + f(x)$ in the form $y = x + 1 + \frac{1}{x-1}$ first as it assists in noting the correct oblique asymptote (diii).

In better responses, students:

- show the ability to obtain the cross-sectional area of a general slice, $\boldsymbol{A = \frac{\sqrt{3}}{4}x^2}$, before obtaining an integral in terms of $\boldsymbol{y}$ for the volume of the solid. Once established, the majority of responses correctly evaluated the integral (a)
- show the ability to differentiate implicitly before solving for $\boldsymbol{\frac{dy}{dx}}$ (bi)
- obtain a quadratic equation in one variable by deducing that $\boldsymbol{y = -2x}$ since $\boldsymbol{\frac{dy}{dx} = 0}$ (bii)
- solving this quadratic equation and substituting the solutions into $\boldsymbol{y = -2x}$ provided the coordinates of the points on the curve where $\boldsymbol{\frac{dy}{dx} = 0}$. The better responses did not substitute the quadratic's solutions into the equation of the curve, an approach that leads to extra and unwanted points (bii)
- evaluate the integral correctly. The majority of successful responses added and subtracted the number 5 into the numerator to rewrite the integrand equal to $\boldsymbol{1 - \frac{5}{x^2+2x+5}}$. Completing the square in the denominator and applying standard integral results complete the integration (c)
- provide a sketch showing the asymptotes and the cusp at the origin (di)
- provide a sketch showing the asymptotes and the turning point at the origin (dii)
- write the equation of the curve $\boldsymbol{y = x + f(x)}$ in the form $\boldsymbol{y = x + 1 + \frac{1}{x-1}}$ which aids in producing the correct sketch with oblique asymptote $\boldsymbol{y = x + 1}$ (diii).

Areas for students to improve include:

- careful development of the ∂V expression (a)
- simplification of algebraic expressions involving squares (a)

- practising implicit differentiation for a variety of curves written implicitly (bi)
- practising their algebraic skills as a significant number did not solve $2x + y = 0$ correctly, instead getting $y = 2x$ (bii)
- checking their solutions; in this case by substituting the coordinates of the points found into both the equation of the curve and into the equation $\frac{dy}{dx} = 0$, to ensure they have the correct solution (bii)
- making use of formulae given in examinations, such as standard integral results (c)
- clearly showing key features of graphs, such as cusps and asymptotes (di)
- clearly showing key features of graphs, such as turning points and asymptotes (dii)
- clearly showing key features of graphs, such as turning points and vertical and oblique asymptotes (diii).

Question 13

Skills addressed:

- simplifying algebraic expressions before integrating (a)
- understanding that i is not part of the modulus of a complex number (bi)
- writing a concluding statement with the answer as specified in the question (bii)
- considering the significance of the statement that "the particle remained in contact with the horizontal surface" in their solution (c)
- understanding the difference between the Normal force and the Tension of the string (c)
- simplifying expressions inside brackets before squaring the expression to reduce the complexity of the solution which may lead to algebraic errors (di)
- simplifying expressions to a point where it is clear $SP = RQ$ (di)
- recognising the significance of part (i) in their solution to this part (dii).

In better responses, students:

- develop an expression for $\partial \boldsymbol{V}$ from a slice which leads to the integral
- $4\pi \int_0^1 (1-x)\sqrt{x}(1-x)\, dx$ and then integrated correctly (a)
- find $\sqrt{(\mathbf{1} - \cos 2\boldsymbol{\theta})^2 + (\sin 2\boldsymbol{\theta})^2}$ and simplify for the correct expression (bi)
- write $\mathbf{2} \sin \boldsymbol{\theta} (\sin \boldsymbol{\theta} + \boldsymbol{i} \cos \boldsymbol{\theta})$ in mod-arg form and achieve the correct answer (bi)
- use $\arg(\boldsymbol{z}) = \tan^{-1}\left[\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right]$ to gain $\tan^{-1}(\cot \boldsymbol{\theta})$ and hence $\arg \boldsymbol{z} = \left(\frac{\pi}{2} - \boldsymbol{\theta}\right)$ (bii)
- use the mod-arg answer from part (i) to express as $\mathbf{2} \sin \boldsymbol{\theta} (\cos(\frac{\pi}{2} - \boldsymbol{\theta}) + \boldsymbol{i} \sin(\frac{\pi}{2} - \boldsymbol{\theta}))$ (bii)
- find correct vertical and horizontal forces, eliminate $\boldsymbol{T}$ to find an expression in $\boldsymbol{N}$, then state $\boldsymbol{N} \geq \mathbf{0}$ and solved to gain the correct inequality (c)
- find the points $\boldsymbol{S}$ and $\boldsymbol{R}$ then use the distance formula to show $\boldsymbol{PS} = \boldsymbol{QR}$ (di)
- find the midpoint of $\boldsymbol{PQ}$ and $\boldsymbol{SR}$ and show they had a shared midpoint (di)
- find the midpoint of the $\boldsymbol{x}$ and $\boldsymbol{y}$-intercepts of the tangent and show that it satisfies the locus equation (dii).

Areas for students to improve include:

- careful development of the ∂V expression for cylindrical shells (a)
- simplifying algebraic expressions involving squares and square roots (a)

- using the most efficient trigonometric substitutions to simplify expressions in order to reduce errors (bi)
- using different variables for different angles. Solutions such as $tan\theta = cot\theta$ should have read $\tan(\arg(z)) = \cot\theta$ or $\tan(\alpha) = \cot\theta$ (bii)
- learning and recognising the complementary angle results (bii)
- not quoting standard circular motion formulas but considering the context of the question (c)
- setting out solutions that involve inequalities with clear explanations of why the inequality has been introduced (c)
- learning the correct formulas for distance and midpoint (di)
- taking care with algebraic simplification (di)
- looking for the most efficient method to gain an answer. Students who solved $xy = c^2$ and $y = tx - at^2$ simultaneously then used the quadratic formula found it difficult to obtain the correct result (dii).

Question 14

Skills addressed:

- changing the limits of integrals when making a substitution (a)
- rearranging compound fractions inside integral signs (a)
- using $v\frac{dv}{dx}$ to find an expression for x, evaluating the constant using the initial conditions then rearranging the expression using logarithm rules to gain the correct expression (b)
- knowing the purpose of finding a reduction formula (cii)
- understanding the language of probability, in particular the concept of 'at least' (dii).

In better responses, students:

- substitute *t* for θ and were able to integrate the expression correctly (a)
- make sure they use the correct choices for $\boldsymbol{u}$ and $\boldsymbol{\frac{dv}{dx}}$ in integration by parts questions, that will lead to the required expression (ci)
- use integration by parts to gain an expression in x^{n-1} then split $x^{n-1}(x+3)^{\frac{3}{2}}$ into $x^n(x+3)^{\frac{1}{2}} + 3x^{n-1}(x+3)^{\frac{1}{2}}$ to gain a correct expression (ci)
- use the reduction formula then calculated I_0 (cii)
- use clear and concise language, clearly interpreting $2 \times \left(\frac{1}{3}\right)^n$ as 'either *A* or *B* winning' every game and $\left(\frac{2}{3}\right)^n$ as '*C* never wins', to gain the correct expression (dii)
- look at previous parts to see how they relate to the question (diii)
- consider the probability of one person winning every game (part i) and the probability of one person never winning (part ii) then simplify their answer to give correct expression (diii).

Areas for students to improve include:

- taking care when manipulating the constant a in the integration of expressions of the form $\int_0^1 \frac{2}{1+at^2}\,dt$ (a)
- making sure they include $+c$ when integrating (b)

- reading the initial conditions in mechanics questions more carefully (b)
- choosing the correct expression of acceleration to use in different situations (b)
- taking care with the limits of integration (ci)
- following through algebra steps carefully from one line to the next so as not to lose negative signs, n or fractions (ci)
- taking care with negative signs (cii)
- using the reduction formula down to I_0, not stopping at I_1 cii)
- writing clear and concise answers, especially when the question is only worth 1 mark (dii)
- using correct reasoning to obtain the given answer (diii).

Question 15

Skills addressed:

- application of De Moivre's theorem and use of the binomial theorem to expand an expression (bi)
- knowledge of trigonometric identities and work on algebraic manipulation skills (bii)
- knowledge the factorisation for $x^n - 1$ (ci)
- recognising the need for cases to be considered in order to progress to the solution in a correct manner (cii)
- substituting for x and then carefully work through the algebraic steps involved (cii).

In better responses, students:

- recognise that the parametric coordinates of Q are equal to $\left(a\cos\left(\theta + \frac{\pi}{2}\right), b\sin\left(\theta + \frac{\pi}{2}\right)\right)$ to show that Q has coordinates $(-a\sin\theta, b\cos\theta)$ (ai)
- first establish the x-coordinate of Q. The y-coordinate of Q is then found by substituting the x-coordinate into the equation of the ellipse and solving for y, or more simply, by multiplying the x-coordinate by $\frac{b}{a}$ (ai)
- show that $\tan\angle POQ'$ is equal to $\frac{2ab}{(a^2-b^2)\sin 2\theta}$ and that its minimum value occurs when $\sin 2\theta$ is a maximum (aii)
- use the formula for the angle between two straight lines to find $\tan\angle POQ'$ (aii)
- use sigma notation to provide the binomial expansion of $(\cos\theta + i\sin\theta)^8$ before applying De Moivre's theorem (bi)
- carefully show all working in their solutions, dividing each term of the given expression for $\sin 8\theta$ by $2\sin\theta\cos\theta$ before replacing each occurrence of $\cos^2\theta$ with $1 - \sin^2\theta$, followed by expanding and collecting like terms (bii)
- establish the equivalence of the "left hand side" with the "right hand side" or vice versa (ci)
- recognise that, in order to prove the result using the identity in part (c)(i), separate cases for x (such as $x < 1, x \geq 1$) need to be considered (cii)
- use the result in part (c)(ii), replacing x with $\frac{a}{b}$ before completing the proof (cii).

Areas for students to improve include:

- providing all necessary working in solving problems that require a stated result to be shown or proven (ai)
- providing all necessary working in solving problems that require a stated result to be shown or proven (aii)

- correctly applying fundamental formulae such as the formula for the angle between two straight lines (aii)
- eliminating basic errors in algebraic manipulation (bi)
- clearly showing all necessary steps in their responses (bii)
- reviewing their solution if they find they do not reach the given result (bii)
- showing exactly which terms cancel when expanding and simplifying expressions involving many terms (ci)
- providing justification for key assertions in mathematical proofs (cii)
- working on algebraic skills and using index laws/properties correctly (cii).

Question 16

Skills addressed:

- presenting a proof using mathematical induction (a)
- applying relevant skills to show the base case and the deductive step (a)
- proving that two triangles are similar (bi)
- using the ratios of corresponding sides of similar triangles and applying this in the construction of a proof (bi)
- connecting parts of previous solutions to a new problem. This part related to part (i) and the information given in the question that $\frac{BC}{DE} = \frac{CA}{FG} = \frac{BA}{HI} = \sqrt{2}$. (bii)
- manipulating the expansion of a quadratic expression (ci)
- using the relationship between the roots and coefficients of a polynomial expression (ci)
- changing the subject of an equation: expressing α in terms of x (cii)
- recognising that the desired result is the negative of the constant term in the cubic equation whose roots are $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$ and $(\beta - \gamma)^2$ (cii)
- knowing and applying the fundamental theorem of algebra and the conjugate root theorem (ciii)
- using the results of the previous parts to establish the proof (ciii).

In better responses, students:

- work from the RHS to show that $\left(x^{3^k} - 1\right)\left(x^{3^{2\times k}} + x^{3^k} + 1\right) = x^{3^{k+1}} - 1$ (a)
- work from the LHS and express $x^{3^{k+1}} - 1$ as $x^{3\times 3^k} - 1 = \left(x^{3^k}\right)^3 - 1$ and then use the difference of two cubes to obtain the RHS (a)
- identify the triangles required to obtain a connection between BC, BF and HC, before using parallelograms to complete the proof (bi)
- use the fact that parallel lines preserve the ratio of intercepts on transversals (bi)
- use the result from part (i) to show that $YZ = DE - 2DY$ and $DY = BC - HC$ before using the ratios connecting DE and HC to BC (bii)
- obtain the connection between $(\beta + \gamma)^2$ and $(\beta - \gamma)^2$ and combine these with the result $\alpha\beta\gamma = -q$ to achieve the desired outcome (ci)
- use one of a number of valid approaches to achieve the appropriate result (ci)
- use either $\alpha = \frac{3p}{x+p}$ which was substituted into $\alpha^3 + p\alpha + q = 0$, or $\alpha^3 = -p\alpha - q$ which was substituted into $\frac{1}{\alpha\beta\gamma}(\alpha^3 + 4q)(\beta^3 + 4q)(\gamma^3 + 4q)$ to achieve the desired result (cii)

- realise that the cubic equation with real coefficients could have all real roots, or one real root and 2 complex roots that were conjugates of each other (ciii)
- find the expression for the product of the y-coordinates of the stationary points and show that it was negative (ciii).

Areas for students to improve include:

- practising the application of index laws when multiplying terms and when raising terms to a power to eliminate errors such as $(x^{3^k})^3 = x^{3^{k+1}}$ and not x^{3^k+3} (a)
- practising algebraic manipulation of complex expressions. For example, recognising that $\left(x^{3^k}\right)^3 - 1 = \left(x^{3^k} - 1\right)\left(x^{3^{2\times k}} + x^{3^k} + 1\right)$ is a case of factorising the difference of two cubes (a)
- understanding that geometric claims must be supported with deductive reasoning as students: used ratios of corresponding sides in similar triangles without proof of similarity; claimed a ratio of lengths without proof; claimed equality of intervals without proof (bi)
- practising how to construct geometrical proofs (bi)
- practising algebraic manipulation involving fractions and ratios as students were using incorrect ratios and incorrect interval lengths (bii)
- making use of the relationships between the roots and coefficients of polynomials and connecting this to the question (ci)
- using an efficient approach, rather than changing the LHS and RHS of the given equation and making errors in the in the algebraic manipulation required (ci)
- expanding and then simplifying complex algebraic expressions, as students were attempting to multiply expressions like $(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2$ or $(\alpha^2+\frac{4q}{\alpha})\ (\beta^2+\frac{4q}{\beta})(\gamma^2+\frac{4q}{\gamma})$ and were unable to progress to the solution (cii)
- practising solving problems involving cubic equations involving roots, as students failed to make use of $\alpha^3+p\alpha+q=0$ to enable an effective substitution for α or to simplify $\frac{1}{\alpha\beta\gamma}(\alpha^3+4q)(\beta^3+4q)(\gamma^3+4q)$ (cii)
- practising the use of the conjugate root theorem as many students did not consider the existence of complex conjugate roots (ciii)
- working with the existence of one, two or three distinct real roots using the y-coordinates of the 2 turning points as students managed to find 2 turning points but did not show a knowledge of what to do with them (ciii).