HSC Marking Feedback 2017

Mathematics Extension 2

Written Examination

Question 11

Part (a) (i)

- Common errors:
  - giving a positive value
  - finding the modulus instead of the argument.

Part (a) (ii)

- In better responses, students knew to subtract $\arg(w)$ from $\arg(z)$.
- A common error was attempting to 'realise' the division of the two complex numbers but then failing to find the argument of the answer.

Part (b)

- In most responses, students found the equation of the asymptote of the hyperbola, then correctly found the inverse tangent of their gradient.

Part (c)

- In better responses, students drew a sketch with a ruler and compass, and ensured their sketch covered one-third of a page so as to more clearly show the details.

Part (d)

- In better responses, students used the suggested substitution, found the value of $d\theta$ and the new limits, and then found the correct primitive function and solution.
- Common errors:
  - using a different substitution or another method (integration by parts), making it difficult to find a reasonable primitive function or complete the solution.

Part (e)

- In better responses, students found the height of shells in terms of y then rotated their shells around the x-axis using the radius values of 0 and 2 as required.
- Students that rotated around the y-axis, or attempted methods not involving cylindrical shells found difficulty in obtaining a working integral.

Part (f)

- In better responses, students used the suggested substitution and found the value of dx and the new limits, or used double angles and methods for integrating $\sin^2\theta$.

Question 12

Part (a) (i)

- In most responses, students successfully calculated and simplified $f'(x) = \frac{2e^x}{(e^x+1)^2}$ from which they recognised and stated that $f'(x) > 0$ for all x and so $f(x)$ is increasing.
- A small number of better solutions avoided the derivative and instead wrote $f(x) = 1 - \frac{2}{e^x+1}$ and e^x, hence $e^x + 1$ are increasing (and positive), so $\frac{2}{e^x+1}$ is decreasing, hence $-\frac{2}{e^x+1}$ is increasing, hence $f(x)$ is increasing.

Part (a) (ii)

- Some students had difficulty proving definition of an odd function $f(-x) = -f(x)$ for all x with a clear intermediate step such as:

$$f(-x) = \frac{e^{-x}-1}{e^{-x}+1} = \frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1} = \frac{\frac{1-e^x}{e^x}}{\frac{1+e^x}{e^x}} = \frac{1-e^x}{1+e^x} = -f(x)$$

or

$$f(-x) = \frac{e^{-x}-1}{e^{-x}+1} = \frac{e^{-x}-1}{e^{-x}+1} \times \frac{e^x}{e^x} = \frac{1-e^x}{1+e^x} = -f(x)$$

Part (a) (iii)

- Most responses indicated $f(x) \to 1$ as $x \to \infty$.
- Some based this on a correct limit argument like:

$$f(x) = \frac{e^x-1}{e^x+1} = \frac{1-e^{-x}}{1+e^{-x}} \to \frac{1-0}{1+0} = 1 \quad \text{as} \quad x \to \infty$$

as $e^x \to \infty$ so $e^{-x} \to 0$ as $x \to \infty$

or

$$f(x) = 1 - \frac{2}{e^x+1} \to 1 - 0 = 1 \quad \text{as} \quad e^x + 1 \to \infty \quad \text{so} \quad \frac{2}{e^x+1} \to 0 \quad \text{as } x \to \infty$$

- Others came to the same conclusion based on numerical evidence for large x.

Part (a) (iv)

- Common errors were:
  - sketching only for $x \geq 0$ (and so ignoring $f(x)$ is defined for all x and is an odd function)
  - making a sketch including $x < 0$ which did not satisfy the oddness property.

Part (a) (v)

- Some errors were:
  - converting the limit behaviour of $f(x)$ to the corresponding behaviour for $\frac{1}{f(x)}$ in the graph
  - sketching the graph for the inverse function $f^{-1}(x)$ instead of that for $\frac{1}{f(x)}$.

Part (b)

- A direct factorisation could be:

$$z^2 + (2+3i)z + (1+3i) = z^2 + 2z + 1 + 3iz + 3i = (z+1)^2 + 3i(z+1)$$
$$= (z+1)(z+1+3i),$$

  so the roots are -1 and $-1-3i$.

- There were many errors in complex number arithmetic in the quadratic formula or the equivalent direct 'completing-the-square method'.
- Some students tried to use the formulae for the sum and product of two roots – so if the roots are α and β then: $\alpha + \beta = -(2+3i)$ and $\alpha\beta = 1 + 3i$.
- Some then eliminated one of the roots to get back to the original quadratic equation in α or β.
- Others took the most difficult method of letting $\alpha = a + ib$ and $\beta = c + id$ (with a, b, c and d assumed to be real) and substituting into the above equations, equating real and imaginary parts to obtain 4 equations in a, b, c and d. From there very few could reach the solution.

Part (c)

- This question was very well done with many students producing the standard solution.
- Some students found the integral after an initial substitution: $x = \tan\theta$, performing a similar integration by parts, or by this substitution in the second integral in the standard solution after integration by parts.
- Common errors were:
  - use of an incorrect primitive for $\tan^{-1}(x)$ (usually giving the derivative instead – a correct primitive is not easy so this was not the best choice in using integration by parts)
  - incorrect substitution into a correct integration by parts formula or an incorrect integration by parts formula
  - a small number of students confused $\tan^{-1}(x)$ with $[\tan(x)]^{-1} = \frac{1}{\tan(x)}$.

Part (d) (i)

- A large number of students produced the standard solution.

- A small number of students were familiar with the result but merely restated it without proof.
- A number of students started with: $P(x) = (x-\alpha)^2\, Q(x) + \, R(x)$ with $Q(x)$ and $R(x)$ being polynomials but needed $R(x) = 0$ (for all x) to recover the standard solution and not just assume $R(\alpha) = 0$ and drop off $R'(x)$ in the expression for $P'(x)$ and/or assume $R'(\alpha) = 0$.

Part (d) (ii)

- Many students produced the standard solution.
- A very small number found the solution by repeated long division by $x - 2$ to obtain

$$P(x) = (x-2)(x^3 - \, x^2 - \, x - 2) = \, (x-2)^2(x^2 + x + 1).$$

- A number of students derived by various means a new polynomial that a double root α of $P(x)$ must satisfy. Many also observed $\alpha = 2$ was a root of this new polynomial but they often failed to then show that 2 is in fact a double root of $P(x)$ by either directly showing $(x-2)^2$ divides $P(x)$ or showing the equivalent condition $P(2) = 0$ and $P'(2) = 0$.

Question 13

Part (a)

- This question was well attempted and by a variety of correct methods including one by contradiction.
- A common error was starting from the given inequality, squaring it in an attempt to 'show that', proving nothing.

Part (b) (i)

- Responses demonstrated a good grasp of the relationships between the sum and products of roots and the coefficients of the polynomial.
- Common errors were:
  - neglecting the minus signs
  - confusing the coefficients of the general polynomial with the polynomial $P(x)$.

Part (b) (ii)

- This part was started but only a few students were able to successfully complete.
- A common error was not seeing the connection between part (a) and this part.

Part (c)

- The excellent responses to this part indicated that students were well prepared in this mechanics question on resisted motion.

Part (d)

- A variety of methods were employed with limited success.
- The use of similar triangles and linear expressions were recognised as methods to answer this volume question. Diagrams assisted students' solutions.
- Common errors were:
  - the inability to correctly manipulate the algebra
  - integration and expansion of expressions with fractions.

Part (e)

- Responses indicated that students recognised a multiplication by i was required.
- Common errors were:
  - using the answer and trying to work backwards
  - incorrect labelling of vectors
  - not showing all lines of working and therefore demonstrating how the required result is achieved for a 'show that' question.

Question 14

Part (a) (i)

- Most students were able to correctly found the values of A and B.
- A significant number of responses approached the question via multiplying both sides by $x^4 + 4$ to attain: $16 = (A + 2x)(x^2 - 2x + 2) + (B - 2x)(x^2 + 2x + 2)$and then determine the values of A and B via substitution and the solution of simultaneous equations.
- Students who attempted this part by expanding the result above and equating coefficients were largely successful.

Part (a) (ii)

- Most students were able to correctly derive the result.
- In better responses, students used the result in (a)(i), algebra and techniques of integration to achieve the result. These students showed substitution into the primitive to verify the result.
- Common errors occurred
  - when students determined incorrect values in (a)(i) and then proceeded with the integration, invariably not achieving the stated result
  - from incorrect algebraic manipulation of the given expression
  - when students simply wrote down the correct primitive implied by the result supplied in the question.

Part (a) (iii)

- This part was particularly well done by the majority of students.
- In better responses, students showed the limiting process and fully explained their solution.
- A number of students simply used a calculator to determine an approximate value.

Part (b) (i)

- Most students were able to correctly provide the required solution.
- A common error was stating an incorrect reason such as 'angle in the alternate segment' or 'alternate angles'.

Part (b) (ii)

- Most students were able to correctly provide the required solution.
- Better responses saw the link to (b)(i) and used it in their solution.

Part (b) (iii)

- Many students were able to correctly provide the required proof.
- Better responses saw students using the link to (b)(i) and (b) (ii) and then making use of appropriate circle geometry theorems and properties in their solution.
- A number of approaches to proving the desired result were observed.
- Many were elegant and efficient, while others were far more complicated.
- Students are reminded to provide a labelled diagram as part of their solution.

Part (c) (i)

- A significant number of students were able to correctly provide the required solution after correctly resolving force R into horizontal and vertical components, then making a trigonometric substitution to achieve the stated result.
- Common errors occurred:
  - with the incorrect resolution of force R
  - with incorrect algebraic manipulation.

Part (c) (ii)

- A significant number of students were able to provide the required solution after correctly resolving the forces on the second particle into horizontal and vertical components, then eliminating force T and finally making a substitution to achieve the stated result.
- Common errors occurred with:
  - with the incorrect resolution of the forces acting on the second particle
  - with the manipulation of the derived expressions.

- when students who derived $T\sin\theta - N\cos\theta = mr\omega^2$ and $T\cos\theta + N\sin\theta = mg$ and then manipulated the expressions to achieve: $T\sin\theta = N\cos\theta + mr\omega^2$and $T\cos\theta = mg - N\sin\theta$ progressing to $\tan\theta = \frac{N\cos\theta + mr\omega^2}{mg - N\cos\theta}$, found the algebraic manipulation required to achieve the desired result very difficult. This was also the case for those students who used $\tan\theta = \frac{r}{h}$ in the above expression.

Part (c) (iii)

- Many students were able to provide the required solution after correctly recognising that $N \geq 0$ in (c) (ii), manipulating the expression, making use of trigonometry and solving an inequality.
- Better responses provided justification throughout the solution.
- Common errors occurred:
  - with the incorrect manipulation of the expression from (c) (ii)
  - with incorrect trigonometric results being used.

Question 15

Part (a) (i)

- Students displayed sound knowledge of the integration process to achieve the required result.
- Students used various approaches including:
  - $\int f'(x) \times \{f(x)\}^n dx = \frac{\{f(x)\}^{n+1}}{n+1} + c$
  - integration by parts
  - integration by substitution.
- Less successful responses used incorrect limits of integration after using the integration by substitution approach.

Part (a) (ii)

- Almost all students were able to display some knowledge of the process of integration by parts.
- More capable students were able to recognise that:

$$\frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx = \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)(1-x^2)^{\frac{1}{2}}\,dx$$

  which then enabled them to achieve the required result.

Part (a) (iii)

- More capable students were able to use the identity from part (ii) and their result from part (i) to obtain an appropriate result.

Part (b) (i)

- Almost all students displayed sound knowledge of implicit differentiation. They were then, almost without exception, able to find the equation of the required tangent to the curve.

Part (b) (ii)

- The vast majority of students were able to correctly find the appropriate x- and y-intercepts and then to express $OA + OB$ in terms of c and d.
- More capable students used the fact that $P\ (c, d)$ was on the curve to then show that $OA + OB = a$.
- A considerable number of students, after finding the required intercepts on the axes, proceeded to find the length of AB rather than $OA + OB$.

Part (c) (i)

- The most successful students were able to see that given:

$$\frac{x^2}{a^2} + \frac{y^2}{y^2} = 1 \text{ and } \frac{x^2}{c^2} - \frac{y^2}{d^2} = 1 \text{ then } \frac{x^2}{a^2} + \frac{y^2}{y^2} = \frac{x^2}{c^2} - \frac{y^2}{d^2}.$$

- Many students, who solved the given equations to find ${x_1}^2$ and ${y_1}^2$ also achieved the appropriate result.
- The second approach was, however, often characterised by a lack of attention to detail that resulted in incorrect signs.

Part (c) (ii)

- The vast majority of students were able to obtain the appropriate gradients of the tangents.
- Many students then proceeded to use: $m_1 \times m_2 = -\frac{b^2 x_1}{a^2 y_1} \times \frac{d^2 x_1}{c^2 y_1}$ together with the result obtained in part (i) to show that: $m_1 \times m_2 = -\frac{b^2 + d^2}{a^2 - c^2}$.
- More capable students were then able to use the fact that the foci were equal to show that the tangents at P were perpendicular.
- A number of students incorrectly assumed that because the foci were equal then the eccentricities were equal.

Question 16

Part (a) (i)

- This part was generally well done. The majority of responses correctly applied De Moivre's theorem to determine a^k and a^{-k} before evaluating the sum of $a^k + a^{-k}$.

Part (a) (ii)

- The most successful responses to this part involved finding the sum of all $(2n+1)$ terms in the series, *C*, followed by multiplying by $\frac{1-\overline{\alpha}}{1-\overline{\alpha}}$.
- Many students first split the series into two smaller series before combining each sum.
- Common errors were:
  - incorrectly counting the number of terms in the series
  - not simplifying terms such as $\overline{\alpha}\alpha^n$
  - not providing complete solutions
  - making errors in algebraic manipulation.

Part (a) (iii)

- This part was well done by most students.
- Common errors included:
  - not clearly showing all necessary steps in the solutions
  - having difficulty with simplifying the denominator term $(1-\alpha)(1-\overline{\alpha})$.

Part (a) (iv)

- Most students correctly substituted $\theta = \frac{\pi}{n}$ into the identity in part (iii).
- A common error was not applying basic trigonometric identities correctly to show that $\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n}$ is independent of n.

Part (b)

- This part proved to be surprisingly problematic for many students.
- Common errors included finding just one possible value for a or finding non-numeric values for a in terms of b.

Part (c) (i)

- This part presented challenges to many students.
- The more successful responses considered the two cases provided in the question, while some responses presented sub-cases by considering whether tile D is a different colour or not from tile A.
- All successful responses avoided using fallacious reasoning to arrive at the given answer.

Part (c) (ii)

- This part was attempted by many students.

- The successful responses included clear reasons when presenting the base case (for $n = 1$) and the inductive step.
- Common errors were:
  - treating the base case with $n = 2$ rather than $n = 1$
  - not realizing that the number of ways of choosing colours for any column after the second column is the same as the number of ways of choosing colours for the second column, and applying this to prove the inductive step.

Part (c) (iii)

- Many students substituted $x = 3$ and $n = 5$ into the given formula in part (ii) to find the number of ways of painting the grid if all colours need not be used.
- The correct responses subtracted the number of ways of painting with just two different colours.
- A common error was subtracting the number of ways of painting with exactly two given colours rather than the number of ways of painting with any two colours chosen from three colours.