Notes from the Marking Centre - Mathematics Extension 2

Question 11

(a)(i) This question was attempted well, with most candidates able to calculate the modulus and argument of the complex number. Common problems were:

- neglecting to put the argument into the fourth quadrant, leaving it as a positive
- angle $\frac{\pi}{6}$
- incorrectly converting to $\frac{5\pi}{6}$. while some could convert to the equivalent $\frac{11\pi}{6}$
- not putting into correct 'mod-arg' form, confusing the negative signs.

(a)(ii) This question was attempted well, even if from an incorrect part (i) result. Responses showed that candidates understood how to use De Moivre's Theorem well, and recognized that the imaginary part became zero.

(a)(iii) This question was attempted well by most. Common problems were:

- due to errors in part (i), responses included a negative or fraction answer
- failure to check that the answer was a positive integer, as required.

(b) This question was attempted well by most candidates, with several correct methods being used. Common problems were:

- an incorrect choice of u or dv resulted in a very difficult (or impossible) integral
- an incorrect integration by parts formula was used
- not clearly showing the steps in choosing parts, substituting, and then integrating

(c) Most candidates handled this question well. Common problems were:

- difficulty differentiating the product $2xy$
- making minor algebra errors after the differentiation process was completed.

(d)(i) Responses demonstrated a good grasp of handling the asymptotes and the sections of the graph that were deleted by square rooting the original. Common problems were:

- graphs were often too small, with no evidence of a ruler being used
- not including a scale (with relevant numbers) on each axis
- not labelling asymptotes
- not ensuring the answer is very clearly differentiated from the original when superimposing the new graph over the original.

(d)(ii) Most responses showed the ability to address the reciprocal of all sections of the original graph. Common problems were:

- neglecting to show that the graph was discontinuous at the x intercepts ($x = \pm 3$), or that the graph approached a horizontal asymptote of $y = \frac{-1}{4}$.

(e) In most responses, candidates found the domain of the function. Those who used a graph or systematically checked the corresponding range of the critical values of the domain often obtained the correct solution. Common problems were:

- difficulty with finding the range.

Question 12

(a)(i) Most candidates found the equation of the ellipse.

(a)(ii) The equation needed to calculate the eccentricity was demonstrated in many responses. Common problems were:

- errors when $a^2 = 3$ and $b^2 = 2$ were used.

(a)(iii) The foci were found. Common problems were:

- failure to recognise that the focus is a point and so a set of coordinates was required to complete the solution.

(a)(iv) The equation of a directrix was stated. Common problems were:

- the inability to correctly manipulate the algebra
- not expressing the directrix as an equation and not understanding that it is a line not a point (ie, $x = \pm\frac{9}{\sqrt{5}}$.)

(b)(i) This part was attempted by most candidates with some success. Recognition of the need to use the product rule to differentiate $x\,f(x)$, and what the differentiation of an integral achieves were evident. Common problems were:

- some confusion when trying to use integration by parts in reverse.

(b)(ii) In the majority of responses, candidates achieved the required result. Common problems were:

- not completing the solution, leaving the answer as $x\tan^{-1}x - \int x \frac{1}{1+x^2}\,dx$.

(c)(i) A variety of methods were used to expand (cos θ + i sin θ)4 including binomial theorem, Pascal's triangle and expanding the four brackets.

In some responses, candidates used De Moivre's Theorem and found cos 4 θ and then used the expansion of cos 4 θ to prove the result. However, some candidates did not demonstrate where cos 4 θ came from and just equated it to their expansion.

(c)(ii) In most responses, candidates achieved the required result. Common problems were:

- errors occurring with expansion and the loss of negative signs.

(d)(i) Responses demonstrated knowledge of equations of tangents and normal including the need to find gradients by differentiation. Common problems were:

- errors occurring when the point was not substituted in the differential and $\frac{dy}{dx} = -c^2x^{-2}$ was used in the equation of a line
- not showing all necessary lines of working and demonstrating how the required result is achieved for a 'show that' question.

(d)(ii) This part was attempted using a variety of successful methods.

A concise approach was to find the gradient of the line PQ and make it equal to p2 (the gradient of the normal) followed by a rearrangement to achieve the required result.

Solving simultaneously the equation of the normal and the hyperbola achieved a quadratic and some candidates then used the result of the product or the sum of the roots to complete their proof. In other responses, candidates used the quadratic formula to achieve the required result.

Substitution of the point $Q\left(cq, \frac{c}{q}\right)$ into the equation of the normal was the least successful method as algebraic errors were made when attempting to manipulate the equation to show the required result.

Question 13

(a) Overall this question was not attempted well. Many responses showed an awareness of $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$ but were not able to progress further. Many responses showed an awareness of $\frac{d}{dx}(x \ln x) = x.\frac{1}{x} + \ln x$, often progressing to confirm a minimum value for $\frac{d}{dx}(x \ln x)$.

Far fewer responses produced $\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln f(x))$, where $f(x) = x^x$, established $\frac{f'(x)}{f(x)} = \frac{d}{dx}(x \ln x) = x.\frac{1}{x} + \ln x$ and then progressed to $f'(x) = f(x)(1 + \ln x) = 0$ (or equivalent) for the stationary points.

(b) A great variety of techniques were presented for this question with varying degrees of success. Candidates who approached the problem by establishing cyclic quadrilateral *OCQA*, congruent triangles *OCQ* and *OAQ* or kite *OCQA* generally had a greater amount of success than those attempting to conclude the proof via another path.
The wording of the reasons given at each step of the proof varied greatly in detail and accuracy. The nature of the diagram meant that responses often included many truths from the information gleaned. The more successful responses were direct, with accurate information and correct reasons.

(c)(i) This question was done well. Correct equations for the horizontal and vertical forces were produced in most responses. From here, most responses demonstrated expertise in eliminating the force T_1 to produce the desired result for T_2.

(c)(ii) Many candidates did not attempt this part. Common problems were:

- errors made in algebraic manipulation prevented success in this question.

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Candidates who avoided algebraic errors were generally successful in this section.

(d)(i) This part was not well done. Some responses showed an awareness that if the derivative of $p(x)$ has no real roots then there are no stationary points for $p(x)$. Common problems were:

- failure to obtain $p'(x) = 3ax^2 + 2bx + c$ prevented arriving at the discriminant $\Delta = 4(b^2 - 3ac)$
- focussing on $p(x)$ rather than $p'(x)$ in an attempt to arrive at a correct conclusion

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(d)(ii) Many responses showed a familiarity with the concept of multiplicity. The more successful responses showed an appropriate level of rigour to confirm the result. Common problems were:

- the inability to complete the problem by demonstrating that $P\left(-\frac{b}{3a}\right) = P'\left(-\frac{b}{3a}\right) = P''\left(-\frac{b}{3a}\right) = 0$ implied a root of multiplicity 3 at $x = -\frac{b}{3a}$.

Question 14

(a)(i) This part was done very well as most candidates were able to correctly prove the result.

A significant number of responses approached the question via the use of the 'Reverse Chain Rule' after making an appropriate trigonometric substitution for $\sin^2 x$. Differentiation of the stated result was also used to answer this part. Responses in which candidates approached the question via substitution of $u = \cos x$, after making an appropriate trigonometric substitution for $\sin^2 x$, were largely successful in achieving the desired result.

Responses is which candidates attempted this part via Integration by Parts, were largely unsuccessful.

(a)(ii) This part was done well as in most responses candidates were correctly able to explain the result.

In the better responses, candidates referred to their diagrams and mentioned symmetry about $x = \frac{\pi}{2}$.

An alternative, and less used approach, involved making a substitution of $u = x - \frac{\pi}{2}$ into the stated integral and basing the explanation on the fact that the resulting integrand was an odd function.

In several responses, candidates made use of the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in their explanation. Common problems were:

- not drawing a correct graph for $y = \cos^{2n-1} x$.

(a)(iii) This part was done particularly well by the majority of candidates.

Several candidates made use of the result $\int_0^\pi f(x)\,dx = \int_0^\pi f(\pi - x)\,dx$ in their attempt, resulting in a process that did achieve the answer without the use of Integration by Parts. Common problems were:

- not correctly applying the Integration by Parts process, with the most common error in this case being:

$$I = \int_0^\pi x \sin^3 x dx \qquad u = x \qquad \frac{dv}{dx} = \sin^3 x$$

$$\frac{du}{dx} = 1 \qquad v = -\cos x + \frac{1}{3}\cos^3 x$$

$$\therefore I = \left[-x \times \cos x + \frac{1}{3}\cos^3 x \right]_0^\pi - \int_0^\pi (-\cos x + \frac{1}{3}\cos^3 x)dx$$

- Responses in which candidates approached this part via $I = \int_0^\pi x \sin^3 x dx = \int_0^\pi x \sin x \sin^2 x dx = \int_0^\pi x \sin x (1 - \cos^2 x) dx$ and who repeatedly used Integration by Parts were largely unsuccessful in progressing to the answer.
- using incorrect limits of integration, incorrect expressions for the volume, and rotation around the x – axis.

(b) (i) This part was done well by a substantial number of candidates. The better responses made use of the substitution of $x = \tan\theta$, although a number of candidates used $x = \cot\theta$ as the substitution and were successful in obtaining the required answer. Common problems were:

- difficulty completing the solution by trying to establish a recurrence relationship for the given integral
- errors when attempting to use a trigonometric substitution other than the preferred ones, using an algebraic substitution or making use of an incorrect recurrence relation.

(b)(ii) This part was done well by a substantial number of candidates. Several candidates determined the value of I_2 and then added this to I_0 to obtain the result.

(b)(iii) This part was generally done well by those candidates who were able to see the link from part (b) (ii) In the better responses, candidates made use of the ideas of the previous two parts. These candidates determined an expression for $I_4 + I_2$ or $I_0 + I_2 + I_4$ or $I_0 + 2I_2 + I_4$ and were able to progress to the answer successfully.

Other approaches involved using a division transformation, substitution of $x = \tan\theta$, and the use of Integration by Parts or the generation and use of a recurrence relation. Common problems were:

- incorrect algebraic manipulation of the expression for I_4
- incorrect use of Integration by Parts
- the incorrect generation and use of a recurrence relation.

(c) This part was not attempted well. In better responses, candidates were able to rearrange and factorise the given expression, then by considering the two cases $0 \le x < 1$ and $x \ge 1$, they were able to justify the validity of the inequality.

Proof by contradiction was a popular method of solution, as was squaring both sides and rearranging to obtain a cubic which was then investigated.

It was evident that many candidates tried many strategies in an attempt to prove this question. Common problems were:

- the investigation of $x \ge 0$ only
- incorrect algebraic manipulation of the given inequality
- insufficient justification provided
- the reliance upon often incorrectly drawn graphs to provide justification.

Question 15

(a) Most candidates achieved some level of success in this part.

Many candidates found the coefficients of the required cubic equation by evaluating $\alpha^2 + \beta^2 + \gamma^2$, $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ and $\alpha^2\beta^2\gamma^2$.

Many other candidates first derived the equation $t\sqrt{t} - 3\sqrt{t} + 1 = 0$ after making the substitution $t = x^2$, although a significant number of responses failed to correctly convert this to a cubic equation.

(b)(i) This part was generally done well.

Successful responses started with the equation $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-\mu^2}{x^2}$.

Some candidates, however, made errors in integrating this equation.

The best responses included reasoning for why the velocity is negative by considering the initial direction of motion and negative acceleration of the particle.

(b)(ii) Typically, candidates integrated $\frac{dt}{dx}$, although a common problem was to not correctly identify the limits of integration *b* and *d*.

(b)(iii) This part proved to be challenging. Successful responses involved replacing *d* with 0 in the expression $t = \frac{1}{\mu}\sqrt{\frac{b}{2}}\left(\sqrt{bd - d^2} + \cos^{-1}\sqrt{\frac{d}{b}}\right)$ given in part (b) (ii). Common problems were:

- errors made in evaluating $\cos^{-1}(0)$
- replacing b with d in the given expression for t.

(c)(i) This part was done well. Most candidates correctly decomposed the algebraic fraction into partial fractions of the form $\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+2}+\frac{D}{x+3}$.

Various methods were used to find the constants A, B, C and D, the least successful of which involved equating coefficients of powers of x.
In some responses, candidates added the given partial fractions to prove the identity.

(c)(ii) Successful responses involved rearranging the right-hand side of the identity with a common denominator, or multiplying both sides by $(x+k)$, before replacing x with $-k$ to find an expression for a_k. Common problems were:

- verifying that the identity was true for $n=3$ (by referring to part (c) (i)) and then erroneously declaring the identity to be true for all positive integer values of n.
- not producing sufficient working to support the response

(c)(iii) In many responses candidates successfully substituted $x=1$ in the identity of part (c) (ii). Some successful responses began with integrating the binomial expansion of either $(1+x)^n$ or $(1-x)^n$.
Common problems were:

- mistakenly using the limiting sum formula for a geometric series
- presenting $\frac{1}{n+1}$ as the final answer and omitting to take the limit with n tending to infinity.

Question 16

(a)(i) The majority of candidates were able to consider the real and imaginary parts of $1+z+w=0$ to obtain $1+\cos\theta+\cos\alpha=0$ and $\sin\theta+\sin\alpha=0$.

In better responses, candidates were then able to deduce $\theta=-\alpha$, leading to $\cos\theta=-\frac{1}{2}$; to solve the trigonometric equation to find the solutions $\theta=\frac{2\pi}{3}\ or -\frac{2\pi}{3}$; and to use the fact that $|z|=|w|=1$ to achieve the required result.

In the better responses in which the vector approach was taken, candidates provided either diagrammatic evidence to support their answer or showed that $|z-1|=|w-1|=|z-w|$.
Common problems were:

- not being able to correctly solve $\cos\theta=-\frac{1}{2}$, with the most common incorrect answers being $\theta=\frac{\pi}{3}$ or $\theta=\frac{\pi}{6}$.
- having taken the vector approach, .not providing any further supporting evidence after stating that since $1+z+w=0$ then $1, z$ and w were the vertices of an equilateral triangle

(a)(ii) The majority of candidates were able to see that there was some connection between parts (i) and (ii). However, they were unable to provide supporting evidence.

In the better responses, candidates realised that either multiplying or dividing by $2i$, enabled them to relate their answer to part (i).

In another approach adopted in better responses, candidates obtained equations similar to those in part (a) (i) ie, $1+\sin\theta+\sin\alpha=0$ and $\cos\theta+\cos\alpha=0$, that enabled them to show that $\theta=\frac{7\pi}{6}$ and $\alpha=-\frac{\pi}{6}$ and that the sides of the triangle are of length $2\sqrt{3}$. Common problems were:

- not recognising that there had to be a rotation of $\frac{\pi}{2}$ and an enlargement by a factor of 2.

(b)(i) The most successful approach was to recognise that $v = \left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)u$ and then to show the required result.

In some better responses, candidates realised that $0 + u + v = 0$ and hence that $u = -v$. On raising both sides to the power of 3 they were able to obtain $u^3 + v^3 = 0$ leading to $(u + v)(u^2 - uv + v^2) = 0$ and hence the required result.

(b)(ii) The majority of candidates were successfully able to obtain the appropriate answers. The most common approach was to take a non-zero complex number and rotate it through $\frac{\pi}{3}$ to obtain the other non-zero complex number.

(c)(i) and (ii) Many candidates did not attempt these parts. However, in the better responses, candidates were able to succinctly provide the appropriate reasoning. Common problems were:

- not providing the appropriate reasoning as to why the answer given in the question was correct or not using appropriate language to reason the case

(c)(iii) In many responses, candidates were successful in using the result obtained in part (c) (ii) to achieve the required result. Common problems were:

- introducing $D(n-3)$ and $D(n-4)$ by finding expressions for $D(n-1)$ and $D(n-2)$.

(c)(iv) This question part was challenging, with only some candidates able to successfully argue the case, enabling them to deduce that $D(n) - nD(n-1) = (-1)^n$.

(c)(v) Many candidates simply used the scaffold used in mathematical induction without providing any proof.

In the better responses, candidates used the result from part (c) (iv), that is, stating that $D(k+1) = (k+1)D(k) + (-1)^{k+1}$ and then rewriting, $(-1)^{k+1}$ as $(k+1)! \times \frac{(-1)^{k+1}}{(k+1)!}$ in their approach leading to a successful proof.