Mathematics Extension 1

HSC Marking Feedback 2018

Question 11

Skills addressed:

- applying the factor theorem or long division of polynomials to prove that $x = 1$ is a zero of the polynomial $P(x)$ (ai)
- finding zeroes of a cubic polynomial by the use of techniques such as long division, substitution into the factor theorem, or the sum and products of roots (aii)
- making use of logarithm and index rules and equation techniques to solve a logarithmic equation (b)
- using trigonometry and trigonometric identities to transform the sum of sine and cosine to the sine of a compound angle (c)
- applying circle geometry theorems, in particular the secant theorem, to generate and solve a quadratic equation and selecting the appropriate solution for the length of the segment (d)
- applying knowledge of functions and restrictions on the domain to find the domain of $f(x) = \frac{1}{4x-1}$ (ei)
- solving an inequality with a variable in the denominator by one of a number of various methods, including multiplying by the square of the denominator, finding critical values and testing, considering options or sketching curves to find the solution (eii)
- applying knowledge of calculus, algebra and arithmetic to evaluate the definite integral $\int_{-3}^{0} \frac{x}{\sqrt{1-x}} dx$ given the substitution $u = 1 - x$ (f).

In better responses, students:

- substitute $x = 1$ using the factor theorem to show that, if $P(1) = 0$, then $x = 1$ is a zero of $P(x)$ (ai)
- use long division to show that $x - 1$ is a factor and hence that $x = 1$ is a zero of $P(x)$ (ai)
- divide the polynomial by the factor $x - 1$ to generate the quotient $x^2 - x - 6$; then factorised the quotient correctly to arrive at the other zeroes $x = 3$ and $x = -2$ (aii)
- use the sum and product of cubic polynomials and the zero from (a)(i) to generate two equations $\alpha\beta \cdot 1 = -6$ and $\alpha + \beta + 1 = 2$ and then solve simultaneously to arrive at the other zeroes $x = 3$ and $x = -2$ (aii)
- use the factor theorem to test $P(3) = 0$ and $P(-2) = 0$ and arrive at the zeroes (aii)
- apply the sum to product logarithm rule to simplify the RHS of the equation and transform correctly from logarithmic format to index format to solve the resulting linear equation (b)
- apply the change of base rule to the terms on the left hand side of the equation, and then multiply by the denominator to solve the question (b)

- make use of the formulas $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}(\frac{b}{a})$ to arrive at values of R and α and express their response in the required form (c)
- equate coefficients with the sine compound angle result and solved for R and α (c)
- use the secant theorem to generate the correct ratios, solve the quadratic equation and select the positive solution as the value of x (d)
- use similar triangles to arrive at the correct ratio for the quadratic equation (d)
- solve correctly for $4x - 1 \neq 0$ (ei)
- multiply by the square of the denominator and then proceed to solve the quadratic inequality and provide the correct solution. This is the most favoured solution method (eii)
- use the graph of the hyperbola, the straight line $y = 1$ and the point of intersection to solve the inequality (eii)
- find the critical points and tested points on either side to reach the correct solution (eii)
- consider options to satisfy the given condition: $4x - 1 > 1$ or $4x - 1 < 0$ (eii)
- substitute correctly for u, changing the bounding limits and placing them in the correct positions, substituting correctly to arrive at the solution (f).

Areas for students to improve include:

- showing the actual substitution line where $x = 1$ in order to prove that $x = 1$ is a zero of $P(x)$ (ai)
- taking care when doing long division and factorising (aii)
- ensuring that they review their work to make sure they have answered the question, as some students left their answer as $P(x) = (x - 1)(x - 3)(x + 2)$ and neglected to provide the other zeroes (aii)
- focusing on converting from logarithmic format to index format. Many students struggled to progress from the step $\log_2(5x - 10) = 3$ to $5x - 10 = 8$ (b)
- being familiar with logarithm laws, as many students did not progress from sum to product (b)
- taking care with algebraic manipulation, as many students made errors when expanding $5(x - 2)$ (b)
- taking care with the memorisation of formulas, if used, and taking care when using the compound angle formulae (c)
- taking care when using special value triangles to correctly solve for the required angle and side (c)
- recognising that the solution is a length and needs to be positive (d)
- showing familiarity with correctly generating the ratio of the intercepts of two secants from an external point (d)
- avoiding erroneously solving $4x - 1 > 0$. This error indicated the need for further review of restrictions on domains (ei)
- avoiding solving the inequality by multiplying both sides by $(4x - 1)$ (eii)
- working on algebraic skills when attempting to multiply by the square of the denominator (eii)
- practising solving quadratic inequalities, as well as avoiding writing that ½ is less than ¼ on the number line when testing points or solving the quadratic inequality (eii)
- avoiding common errors such as placing the limits as: $\int_1^4 \frac{1-u}{\sqrt{u}}(-du)$ instead of the correct $\int_4^1 \frac{1-u}{\sqrt{u}}(-du)$; providing the correct integrand, but using original limits; providing the correct

integrand but leaving off the negative sign from $dx = -du$; incorrectly substituting x as $u - 1$ (f)

- avoiding the incorrect application of index laws when manipulating $\frac{1-u}{\sqrt{u}}$ to $(1-u).u^{-\frac{1}{2}}$ as many students used $(1-u).u^{\frac{1}{2}}$ (f)
- practising manipulation of terms with fractional indices such as $\frac{2}{3}(4)^{\frac{3}{2}}$ (f).

Question 12

Skills addressed:

- transforming a given integrand into a form that is easier to integrate (a)
- learning and applying the standard integral for this type of function (a)
- paying attention to the mark value of the question and using it as a guide to the complexity of solution required (bi)
- recognising the difference between angular velocity and tangential velocity (bii)
- looking at the given information and thinking about how it could be used in the solution (bii)
- correctly applying the standard integral for $\sin^{-1} x$ from the Reference Sheet to determine its derivative (ci)
- recognising that a derivative of zero implies that the primitive function was a constant (cii)
- using a straight edge to draw lines (ciii)
- indicating scale on each axis (ciii)
- understanding binomial probability (d)
- knowing the condition for two lines to be perpendicular (ei)
- having fluency with circle geometry theorems and properties (eii).

In better responses, students:

- use the double angle results and applied them to $\cos^2 3x$ (a)
- handle what is essentially a basic trigonometric relationship and follow by simple differentiation (bi)
- recognise and use related rates (bii)
- successfully integrate $\cos^{-1} x$, which is not included in the Reference Sheet (ci)
- substitute a value into the function to establish the value of the constant, and show the result of the substitution into each term (cii)
- realise that the graph of a constant value is a horizontal line (ciii)
- restrict the domain of the answer to the common domain of the two functions (ciii)
- understand what is meant by the key phrase 'at least' (d)
- refer to the Reference Sheet and did not spend time deriving results that were contained on the sheet (ei)
- see the connection with part (i) and make use of the parametric approach to problems and do not waste time repeating their solution to part (i) to show that angle TBS is a right angle (eii)
- successfully apply circle geometry theorems and properties in an 'out of context' problem (eii)

- use the distance formula to find the length of the diameter, stating the reason why ST is a diameter (eiii)
- choose to use Pythagoras' Theorem on triangle SAT, with appropriate reasoning (eiii).

Areas for students to improve include:

- practising the use of standard integrals in a variety of situations and not just the most basic case, particularly if they plan to memorise standard integrals (a)
- avoiding reading too much into the question and trying to incorporate the chain rule too early (bi)
- applying the chain rule correctly to determine $\frac{dh}{dt}$ (bii)
- relating the angular velocity given in the question to $\frac{d\theta}{dt}$ (bii)
- practising using the given integrals in non-standard situations (ci)
- avoiding leaving the integral in standard form, $\frac{1}{\sqrt{a^2-x^2}}$, and not substituting $a = 1$ (ci)
- avoiding writing the integral for $\cos^{-1} x$ as $\frac{1}{\sqrt{a^2+x^2}}$ (ci)
- avoiding the omission of too many steps of the proof, and communicating clearly about how they went from one step to the next (cii)
- realising that when graphing the sum of two functions, the resulting graph is restricted to the common domain of the two functions (ciii)
- understanding the meaning of 'at least' by including, in this case, the probability of ten people finishing in their calculation (d)
- familiarising themselves with the contents of the Reference Sheet and, in this case, not wasting time deriving the equation of the tangent (ei)
- learning correct terminologies and avoiding use of colloquial expressions such as 'bowtie theorem' (eii)
- articulating their reasoning in their answer (eii)
- avoiding the omission of too many lines in the algebraic manipulation in an attempt to show the given result. In a 'show' question it must be clear how one line is obtained from another (eiii).

Question 13

Skills addressed:

- verifying the initial case and set up the statement $S_k + T_{k+1}$ (a)
- recognising the need for the domain restriction in $f(x)$, before establishing the domain for $f^{-1}(x)$ (bi)
- using domain and range, and other graphing knowledge to produce a sketch (bii)
- interchange x and y (biii)
- solving the resultant equation (biii)
- solving $y(t) = 0$ to find the time, and proceeding on to substitute $t = \frac{2Vsin\theta}{g}$ into $x = Vtcos\theta$ (ci)
- substituting $(\frac{\pi}{2} - \theta)$ into an appropriate equation from the problem and re-solving (cii)
- establishing separate range equations in terms of α and β before finding average (ciii)

- recognising the value of the information obtained in parts (i) and (ii) (ciii).

In better responses, students:

- establish $\frac{1-(-3)(-3)^k}{2} = \frac{1-(-3)^{k+1}}{2}$ correctly (a)
- reach a successful conclusion for S_{k+1} (a)
- determine the correct domain as $0 < x \leq \frac{1}{2}$ (bi)
- obtain a correct graph in (bi) coming from the correct domain and range (bii)
- solve the equation by using quadratic formula or completing the square method (biii)
- use the domain restriction correctly to provide final solution (biii)
- complete the proof by using the appropriate double angle identity (ci)
- establish the result from (ci) by using the range determined (cii)
- link the working from the previous two parts to establish equations in terms of α and β (ciii)
- correctly use the given substitution $\beta = \frac{\pi}{2} - \alpha$ to eliminate α and β (ciii).

Areas for students to improve include:

- accurately manipulating algebra (a)
- managing the connection between LHS and RHS effectively (a)
- considering $f^{-1}(x)$ as more than the interchange between x of y only (bi)
- checking the graph with a vertical line test (bii)
- enhancing the diagram with more detail such as the line $x = y$ or the point $(½, 1)$ (bii)
- moving on algebraically from $x = \frac{y}{y^2+1}$ (biii)
- using the domain restriction to establish the final solution $f^{-1}(x) = \frac{1+\sqrt{1-4x^2}}{2x}$ (biii)
- using the given equations (no need to derive) (ci)
- not using the Cartesian equation of the projectile unless necessary (ci)
- considering the aim of this problem and why this proof might be needed or linked to (cii)
- using the trigonometric identities more efficiently and effectively to simplify the algebra (ciii)
- finding an average for the two range equations (ciii).

Question 14

Skills addressed:

- taking care with the Greek letters used in geometry. They were often interchanged (a)
- showing all reasoning in geometry responses (a)
- being clear and thorough when giving binomial expansions (bi)
- using the correct terminology such as ‘equating the correct terms in x^r from both expansions’ (bi)
- expecting that (bii) is connected to part (bi) in this question
- showing appropriate working and not give an unsupported answer (bii)
- clearly stating the appropriate angles (ci)
- being aware of the difference between congruency and similarity (ci)

- matching the correct sides of the similar triangles when the triangles are nested (cii)
- using the required result, $S = \frac{\pi ab^2}{4(2c-a)}$, as a guide (ciii)
- seeking an inequality based on the shapes given (civ)
- attempting later parts of questions despite being unsuccessful earlier on (civ).

In better responses, students:

- quickly determine the 3 main geometrical relationships, namely

  $2\alpha + 2\beta + 2\gamma + 2\delta = 360°$ (angle sum of quadrilateral $ABCD$)

  $\angle PSR = \angle BSC = 180° - (\beta + \gamma)$ (angle sum of ΔBSC)

  $\angle PQR = \angle AQD = 180° - (\alpha + \delta)$ (angle sum of ΔAQD)

  which formed the basis of the proof (a)
- determine the need to find the terms in x^r from the outset of the question (bi)
- simply state that from (bi) the counting could be done in $\binom{23}{4}2^{19}$ ways (bii)
- show the common angle and one other angle for the proof (ci)
- show a simple relationship between the corresponding sides for a correct solution (cii)
- use equivalent expressions for the area of the triangle to show the result (cii)
- realise that the successive radii need to be written in terms of a, b or c so that the areas would also be in these terms (ciii)
- apply appropriate geometric series formula with a correct ratio, typically $r = (1 - \frac{a}{c})^2$ (ciii)
- recognise that the inequality connected the area of the triangle with the area of the quadrants (civ).

Areas for students to improve include:

- seeking to find angles directly related to the cyclic quadrilateral in question (a)
- following the instructions within the question which asked them to copy or trace the diagram, as many did not do this (a)
- recognising the need to search for the coefficients of a particular power of x in an expansion (bi)
- expanding $(2 + x)^n$ and $\big(1 + (1 + x)\big)^n$correctly with the powers in the correct order (bi)
- using the $\binom{n}{r}$ expression in binomial expansions (bi)
- avoiding making huge arithmetic calculations (bii)
- avoiding misinterpreting the question as a probability question (bii)
- using correct angle definitions (ci)
- avoiding mixing the order of sides to produce an incorrect ratio, for example $\frac{x}{a} = \frac{c}{b}$ (cii)
- refraining from using Pythagoras' Theorem, as this leads to an incorrect result (cii)
- refraining from trying to determine the ratio by working back from the result (ciii)
- finding the area of the first and second quadrant, as this would give an indication of the ratio (ciii)
- finding a correct relationship for x_2 and x_1 or x (ciii)
- completing questions involving a comparison of areas to produce an inequality (civ).